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On some multipoint methods arising from optimal in the sense of Kung–Traub algorithms

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Abstract—**In this paper we will examine self-accelerating in terms of convergence speed and the corresponding index of efficiency in the sense of Ostrowski–Traub of certain standard and most commonly used in practice multipoint iterative methods using several initial approximations for numerical solution of nonlinear equations due to optimal in the sense of the Kung–Traub algorithm of order 4, 8 and 16. Some hypothetical iterative procedures generated by algorithms from order of convergence 32 and 64 are also studied (the receipt and publication of which is a matter of time, having in mind the increased interest in such optimal algorithms). The corresponding model theorems for their convergence speed and efficiency index have been formulated and proved.**

Keywords-**solving nonlinear equations; order of convergence; optimal algorithm; efficiency index**

I. Introduction

One of the most basic problems in scientific and engineering applications is to find the solution of a nonlinear equation

$$f(x) = 0. \tag{1}$$

In literature, it is known that the computational efficiency of a method is measured by the concept of the efficiency index $p^{\frac{1}{n}}$, where p is the order of convergence and n is the whole number of functional evaluations per iteration. Subsequently, the maximum efficiency index for Newton's iteration with two functional evaluations is $2^{\frac{1}{2}} \approx 1.414$ [30].

According to the conjecture of Kung and Traub [13], the maximum convergence order of a scheme (without memory) including n evaluations per step is 2^{n-1}.

By taking into account the optimality concept, many authors have tried to build iterative procedures of optimal order of convergence $p = 4$, $p = 8$, $p = 16$.

The recent results of M. Petkovic [20] and M. Petkovic and L. Petkovic [22], Bi, Wu and Ren [2], Geum and Kim [7], Thurkal and Petkovic [29], Wang and Liu [31], Kou, Wang and Sun [12], Chun and Neta [3], Soleymani and Soleymani [24], Soleymani [25], Bi, Ren and Wu [1], Sargolzaei and Soleymani [26], Soleymani and Mousavi [27], Soleymani and Sharifi [28], Ignatova, Kyurkchiev and Iliev [10], M. Petkovic, Neta, L. Petkovic and Dzunic [21] are presented for optimal multipoint methods for solving nonlinear equations. For other results see Dzunic and M. Petkovic [6].

M. Petkovic [20] gives a useful detailed review about computational efficiency of many methods in the sense of Kung–Traub hypothesis.

For other nontrivial methods for solving nonlinear equations see, Kyurkchiev and Iliev [14] and Iliev and

Kyurkchiev [11].

In many natural science tasks, from purely physical considerations, the user of numerical algorithms for solving nonlinear equation (1) knows a set of initial approximations

$$x_1^0, x_2^0, \dots, x_k^0$$

for the root ξ of equation (1).

As an example, regula falsi methods and modifications of Euler–Chebyshev method and Halley method with a lower order of convergence use two or three initial approximations for the root ξ.

In [16], refined conditions of convergence for the difference analogue of Halley method (using three initial approximations) for solving nonlinear equation are given (see, also [32]).

An efficient modification of a finite–difference analogue of Halley method is proposed in [9].

Naturally arises the task of designing and testing multipoint variants of the classical procedures in the light of the achievements over the past five years important theoretical results related to obtaining optimal in the sense of Kung–Traub algorithms.

In this sense the task of detailed refinement of the self-accelerating multipoint methods using several initial approximations become very actual.

II. Main Results

A. Optimal algorithm in the sense of Kung–Traub with order of convergence $p = 4$

We consider the following nonstationary iterative scheme based on the 4-point iteration function in combination with an optimal algorithm in the sense of Kung–Traub with order of convergence $p = 4$:

$$\begin{aligned} x_{2n+1} &= \varphi_1(x_{2n}, x_{2n-1}, x_{2n-2}, x_{2n-3}), \\ x_{2n+2} &= \varphi_2(x_{2n+1}). \end{aligned} \tag{2}$$

It is known that for the error $\epsilon_i = x_i - \xi, \quad i = -3, -2, -1, 0, 1, 2 \dots$; [30] is valid

$$\epsilon_{2n+1} \sim C_1(\xi)\epsilon_{2n}\epsilon_{2n-1}\epsilon_{2n-2}\epsilon_{2n-3}, \tag{3}$$

$$\epsilon_{2n+2} \sim C_2(\xi)\epsilon_{2n+1}^4. \tag{4}$$

Let

$$\begin{aligned} K_9 &= \max\{|C_1(\xi))|, |C_2(\xi)|\}, \\ d_{2n-1} &= K_9^{\frac{1}{3}}|\epsilon_{2n-1}|, \\ d_{2n} &= K_9^{\frac{1}{3}}|\epsilon_{2n}|, \end{aligned}$$

and let $0 < d < 1$, and x_{-3}, x_{-2}, x_{-1} and x_0 be chosen so that the following inequalities

$$\begin{aligned} d_{-3} &= K_9^{\frac{1}{3}}|x_{-3} - \xi| \le d < 1, \\ d_{-2} &= K_9^{\frac{1}{3}}|x_{-2} - \xi| \le d < 1, \\ d_{-1} &= K_9^{\frac{1}{3}}|x_{-1} - \xi| \le d < 1, \\ d_0 &= K_9^{\frac{1}{3}}|x_0 - \xi| \le d < 1 \end{aligned}$$

hold true.

From (3) and (4), we have

$$\begin{aligned} & d_{2n+1} = K_9^{\frac{1}{3}}|\epsilon_{2n+1}| \\ & \le K_9^{\frac{1}{3}}K_9|\epsilon_{2n}||\epsilon_{2n-1}||\epsilon_{2n-2}||\epsilon_{2n-3}| \\ & = K_9^{\frac{1}{3}}|\epsilon_{2n-1}|K_9^{\frac{1}{3}}|\epsilon_{2n}|K_9^{\frac{1}{3}}|\epsilon_{2n-2}|K_9^{\frac{1}{3}}|\epsilon_{2n-3}| \\ & = d_{2n}d_{2n-1}d_{2n-2}d_{2n-3}, \\ & d_{2n+2} = K_9^{\frac{1}{3}}|\epsilon_{2n+2}| \le K_9^{\frac{1}{3}}K_9\epsilon_{2n+1}^4 \\ & = \left(K_9^{\frac{1}{3}}\epsilon_{2n+1}\right)^4 = d_{2n+1}^4. \end{aligned} \tag{5}$$

Evidently, from (5), we find

$$\begin{aligned} & d_1 \le d^4,\ d_2 \le d^{16},\ d_3 \le d^{22},\ d_4 \le d^{88}, \\ & d_5 \le d^{130},\ d_6 \le d^{520},\ d_7 \le d^{760},\ d_8 \le d^{3040}, \\ & d_9 \le d^{4450},\ d_{10} \le d^{17800}. \end{aligned}$$

Our results concerning the order of convergence generated by (2) are summarized in the following theorem.

Theorem A. *Assume that the initial approximations x_0, x_{-1}, x_{-2}, x_{-3} are chosen so that $d_{-3} \le d$, $d_{-2} \le d$, $d_{-1} \le d < 1$ and $d_0 \le d < 1$. Then for the error of the sequences $\{x_{2n+1}\}_{n=0}^{\infty}$ and $\{x_{2n+2}\}_{n=0}^{\infty}$ determined by (2), we have*

$$\begin{aligned} d_{2n-1} &\le d^{\tau_{2n-1}}, \\ d_{2n} &\le d^{\tau_{2n}}, \end{aligned} \tag{6}$$

where

$$\tau_{m+4} = 5\tau_{m+2} + 5\tau_m, \quad m = 1, 2, \dots \tag{7}$$

and the order of convergence of the iteration (2) is

$$\tau = \frac{5 + 3\sqrt{5}}{2}.$$

Proof. It is well known that the recursion:

$$\gamma_{i+1} = \sum_{j=1}^{n} A_j\gamma_{i-j+1}, \quad i = n-1, n-2, \dots,$$

(for any initial conditions) corresponds to the characteristic polynomial:

$$\rho^n = \sum_{j=1}^{n} A_j \rho^{n-j}.$$

In our case, for the recursion

$$\tau_{m+4} = 5\tau_{m+2} + 5\tau_m,$$

the characteristic polynomial is of the type

$$\rho^2 - 5\rho - 5 = 0. \qquad (8)$$

Equation (8) has the roots:

$$\rho_1 = \frac{5+3\sqrt{5}}{2}, \quad \rho_2 = \frac{5-3\sqrt{5}}{2}.$$

From the general iterative theory [30], (see, also [8]) it follows that the order of convergence of the iteration procedure, defined by (2) is given by the only real root of equation (8) with magnitude greater than 1. On the other hand,

$$|\epsilon_{2n+1}| \le K_9^{-\frac{1}{3}} d_{2n+1}, \quad |\epsilon_{2n+2}| \le K_9^{-\frac{1}{3}} d_{2n+2},$$

and consequently we can conclude that the order of convergence of iteration (2) is

$$\tau = \frac{5+3\sqrt{5}}{2} \approx 5.8541...$$

Thus, the theorem is proven.

B. Optimal algorithm in the sense of Kung–Traub with order of convergence $p = 8$

We consider the following nonstationary iterative scheme based on the 4-point iteration function in combination with an optimal algorithm in the sense of Kung–Traub with order of convergence $p = 8$:

$$\begin{aligned} x_{2n+1} &= \varphi_1(x_{2n}, x_{2n-1}, x_{2n-2}, x_{2n-3}), \\ x_{2n+2} &= \varphi_3(x_{2n+1}). \end{aligned} \qquad (9)$$

For the error $\epsilon_i = x_i - \xi, \quad i = -3, -2, -1, 0, 1, 2 \ldots$; [30], [23] is valid

$$\epsilon_{2n+1} \sim C_1(\xi)\epsilon_{2n}\epsilon_{2n-1}\epsilon_{2n-2}\epsilon_{2n-3}, \qquad (10)$$

$$\epsilon_{2n+2} \sim C_3(\xi)\epsilon_{2n+1}^8. \qquad (11)$$

Let

$$\begin{aligned} K_{10} &= \max\{|C_1(\xi))|, |C_3(\xi)|\}, \\ d_{2n-1} &= K_{10}^{\frac{3}{17}} |\epsilon_{2n-1}|, \\ d_{2n} &= K_{10}^{\frac{7}{17}} |\epsilon_{2n}|, \end{aligned}$$

and let $d > 0$, and x_{-3}, x_{-2}, x_{-1} and x_0 be chosen so that the following inequalities

$$\begin{aligned} d_{-3} &= K_{10}^{\frac{3}{17}} |x_{-3} - \xi| \le d < 1, \\ d_{-2} &= K_{10}^{\frac{7}{17}} |x_{-2} - \xi| \le d < 1, \\ d_{-1} &= K_{10}^{\frac{3}{17}} |x_{-1} - \xi| \le d < 1, \\ d_0 &= K_{10}^{\frac{7}{17}} |x_0 - \xi| \le d < 1 \end{aligned}$$

hold true.

From (10) and (11), we have

$$\begin{aligned} & d_{2n+1} = K_{10}^{\frac{3}{17}} |\epsilon_{2n+1}| \\ & \le K_{10}^{\frac{3}{17}} K_{10} |\epsilon_{2n}||\epsilon_{2n-1}||\epsilon_{2n-2}||\epsilon_{2n-3}| \\ & = K_{10}^{\frac{3}{17}} K_{10}^{\frac{3}{17}+\frac{7}{17}+\frac{7}{17}} |\epsilon_{2n}||\epsilon_{2n-1}||\epsilon_{2n-2}||\epsilon_{2n-3}| \\ & = d_{2n} d_{2n-1} d_{2n-2} d_{2n-3}, \\ & d_{2n+2} = K_{10}^{\frac{7}{17}} |\epsilon_{2n+2}| \\ & \le K_{10}^{\frac{7}{17}} K_{10} \epsilon_{2n+1}^8 = \left(K_{10}^{\frac{3}{17}} \epsilon_{2n+1}\right)^8 = d_{2n+1}^8. \end{aligned} \qquad (12)$$

From (12), we find

$$\begin{aligned} & d_1 \le d^4,\ d_2 \le d^{32},\ d_3 \le d^{38},\ d_4 \le d^{304}, \\ & d_5 \le d^{378},\ d_6 \le d^{3024},\ d_7 \le d^{3744},\ d_8 \le d^{29952}, \\ & d_9 \le d^{37098},\ d_{10} \le d^{296784}. \end{aligned}$$

Our results concerning the order of convergence generated by (9) are summarized in the following theorem.

Theorem B. *Assume that the initial approximations x_0, x_{-1}, x_{-2}, x_{-3} are chosen so that $d_{-3} \le d$, $d_{-2} \le d$, $d_{-1} \le d < 1$ and $d_0 \le d < 1$. Then for the error of the sequences $\{x_{2n+1}\}_{n=0}^{\infty}$ and $\{x_{2n+2}\}_{n=0}^{\infty}$ determined by (9), we have*

$$\begin{aligned} d_{2n-1} &\le d^{\tau_{2n-1}}, \\ d_{2n} &\le d^{\tau_{2n}}, \end{aligned} \qquad (13)$$

where

$$\tau_{m+4} = 9\tau_{m+2} + 9\tau_m, \quad m = 1, 2, \ldots \qquad (14)$$

and the order of convergence of the iteration (9) is

$$\tau = \frac{3\left(3+\sqrt{13}\right)}{2}.$$

Proof. In our case, for the recursion

$$\tau_{m+4} = 9\tau_{m+2} + 9\tau_m,$$

the characteristic polynomial is of the type

$$\rho^2 - 9\rho - 9 = 0. \qquad (15)$$

Equation (15) has the roots:

$$\rho_1 = \frac{3\left(3+\sqrt{13}\right)}{2}, \quad \frac{3\left(3-\sqrt{13}\right)}{2}.$$

From the general iterative theory it follows that the order of convergence of the iteration procedure, defined by (9) is given by the only real root of equation (15) with magnitude greater than 1. On the other hand,

$$|\epsilon_{2n+1}| \le K_{10}^{-\frac{3}{17}} d_{2n+1}, \quad |\epsilon_{2n+2}| \le K_{10}^{-\frac{7}{17}} d_{2n+2},$$

and consequently we can conclude that the order of convergence of iteration (9) is

$$\tau = \frac{3\left(3+\sqrt{13}\right)}{2} \approx 9.9083...$$

Thus, the theorem is proven.

C. Optimal algorithm in the sense of Kung–Traub with order of convergence $p = 16$

We consider the following nonstationary iterative scheme based on the 4-point iteration function in combination with an optimal algorithm in the sense of Kung–Traub with order of convergence $p = 16$:

$$\begin{aligned} x_{2n+1} &= \varphi_1(x_{2n}, x_{2n-1}, x_{2n-2}, x_{2n-3}), \\ x_{2n+2} &= \varphi_4(x_{2n+1}). \end{aligned} \qquad (16)$$

It is known that for the error $\epsilon_i = x_i - \xi, \quad i = -3, -2, -1, 0, 1, 2\ldots$; [30] is valid

$$\epsilon_{2n+1} \sim C_1(\xi)\epsilon_{2n}\epsilon_{2n-1}\epsilon_{2n-2}\epsilon_{2n-3}, \qquad (17)$$

$$\epsilon_{2n+2} \sim C_4(\xi)\epsilon_{2n+1}^{16}. \qquad (18)$$

Let

$$\begin{aligned} K_{11} &= \max\{|C_1(\xi))|, |C_4(\xi)|\}, \\ d_{2n-1} &= K_{11}^{\frac{1}{11}}|\epsilon_{2n-1}|, \\ d_{2n} &= K_{11}^{\frac{5}{11}}|\epsilon_{2n}|, \end{aligned}$$

and let $d > 0$, and x_{-3}, x_{-2}, x_{-1} and x_0 be chosen so that the following inequalities

$$\begin{aligned} d_{-3} &= K_{11}^{\frac{1}{11}}|x_{-3} - \xi| \le d < 1, \\ d_{-2} &= K_{11}^{\frac{5}{11}}|x_{-2} - \xi| \le d < 1, \\ d_{-1} &= K_{11}^{\frac{1}{11}}|x_{-1} - \xi| \le d < 1, \\ d_0 &= K_{11}^{\frac{5}{11}}|x_0 - \xi| \le d < 1 \end{aligned}$$

hold true.

From (17) and (18), we have

$$\begin{aligned} & d_{2n+1} = K_{11}^{\frac{1}{11}}|\epsilon_{2n+1}| \\ & \le K_{11}^{\frac{1}{11}} K_{11}|\epsilon_{2n}||\epsilon_{2n-1}||\epsilon_{2n-2}||\epsilon_{2n-3}| \\ & = K_{11}^{\frac{1}{11}} K_{11}^{\frac{1}{11}+\frac{1}{11}+\frac{1}{11}}|\epsilon_{2n}|\epsilon_{2n-1}||\epsilon_{2n-2}|\epsilon_{2n-3}| \\ & = d_{2n}d_{2n-1}d_{2n-2}d_{2n-3}, \\ & d_{2n+2} = K_{11}^{\frac{5}{11}}|\epsilon_{2n+2}| \\ & \le K_{11}^{\frac{5}{11}} K_{11}\epsilon_{2n+1}^{16} = \left(K_{11}^{\frac{1}{11}}\epsilon_{2n+1}\right)^{16} = d_{2n+1}^{16}. \end{aligned} \qquad (19)$$

From (19), we find

$$\begin{aligned} & d_1 \le d^4,\ d_2 \le d^{64},\ d_3 \le d^{70},\ d_4 \le d^{1120}, \\ & d_5 \le d^{1258},\ d_6 \le d^{20128},\ d_7 \le d^{22576},\ d_8 \le d^{361216}, \\ & d_9 \le d^{405178},\ d_{10} \le d^{6482848}. \end{aligned}$$

Our results concerning the order of convergence generated by (19) are summarized in the following theorem.

Theorem C. *Assume that the initial approximations x_0, x_{-1}, x_{-2}, x_{-3} are chosen so that $d_{-3} \le d$, $d_{-2} \le d$, $d_{-1} \le d < 1$ and $d_0 \le d < 1$. Then for the error of the sequences $\{x_{2n+1}\}_{n=0}^{\infty}$ and $\{x_{2n+2}\}_{n=0}^{\infty}$ determined by (16), we have*

$$\begin{aligned} d_{2n-1} &\le d^{\tau_{2n-1}}, \\ d_{2n} &\le d^{\tau_{2n}}, \end{aligned} \qquad (20)$$

where

$$\tau_{m+4} = 17\tau_{m+2} + 17\tau_m, \quad m = 1, 2, \ldots \qquad (21)$$

and the order of convergence of the iteration (16) is

$$\tau = \frac{\left(17+\sqrt{357}\right)}{2}.$$

Proof. In our case, for the recursion

$$\tau_{m+4} = 17\tau_{m+2} + 17\tau_m,$$

the characteristic polynomial is of the type

$$\rho^2 - 17\rho - 17 = 0. \qquad (22)$$

Equation (22) has the roots:

$$\rho_1 = \frac{\left(17+\sqrt{357}\right)}{2}, \quad \frac{\left(17-\sqrt{357}\right)}{2}.$$

From the general iterative theory it follows that the order of convergence of the iteration procedure, defined by

(16) is given by the only real root of equation (22) with magnitude greater than 1. On the other hand,

$$|\epsilon_{2n+1}| \le K_{11}^{-\frac{1}{11}} d_{2n+1}, \quad |\epsilon_{2n+2}| \le K_{11}^{-\frac{5}{11}} d_{2n+2},$$

and consequently we can conclude that the order of convergence of iteration (16) is

$$\tau = \frac{\left(17+\sqrt{357}\right)}{2} \approx 17.9472...$$

Thus, the theorem is proven.

D. Optimal algorithm in the sense of Kung–Traub with order of convergence $p = 32$

We consider the following nonstationary iterative scheme based on the 4-point iteration function in combination with an optimal algorithm in the sense of Kung–Traub with order of convergence $p = 32$:

$$\begin{aligned} x_{2n+1} &= \varphi_1(x_{2n}, x_{2n-1}, x_{2n-2}, x_{2n-3}), \\ x_{2n+2} &= \varphi_5(x_{2n+1}). \end{aligned} \tag{23}$$

For the error $\epsilon_i = x_i - \xi, \quad i = -3, -2, -1, 0, 1, 2 \ldots;$ [30] is valid

$$\epsilon_{2n+1} \sim C_1(\xi)\epsilon_{2n}\epsilon_{2n-1}\epsilon_{2n-2}\epsilon_{2n-3}, \tag{24}$$

$$\epsilon_{2n+2} \sim C_5(\xi)\epsilon_{2n+1}^{16}. \tag{25}$$

Let

$$\begin{aligned} K_{12} &= \max\{|C_1(\xi))|, |C_5(\xi)|\}, \\ d_{2n-1} &= K_{12}^{\frac{3}{65}}|\epsilon_{2n-1}|, \\ d_{2n} &= K_{12}^{\frac{31}{65}}|\epsilon_{2n}|, \end{aligned}$$

and let $d > 0$, and x_{-3}, x_{-2}, x_{-1} and x_0 be chosen so that the following inequalities

$$\begin{aligned} d_{-3} &= K_{12}^{\frac{3}{65}}|x_{-3} - \xi| \le d < 1, \\ d_{-2} &= K_{12}^{\frac{31}{65}}|x_{-2} - \xi| \le d < 1, \\ d_{-1} &= K_{12}^{\frac{3}{65}}|x_{-1} - \xi| \le d < 1, \\ d_0 &= K_{12}^{\frac{31}{65}}|x_0 - \xi| \le d < 1 \end{aligned}$$

hold true.

From (24) and (25), we have

$$\begin{aligned} d_{2n+1} &= K_{12}^{\frac{3}{65}}|\epsilon_{2n+1}| \\ &\le K_{12}^{\frac{1}{12}} K_{12}|\epsilon_{2n}||\epsilon_{2n-1}||\epsilon_{2n-2}||\epsilon_{2n-3}| \\ &= K_{12}^{\frac{3}{65}} K_{12}^{\frac{3}{65}+\frac{31}{65}+\frac{31}{65}}|\epsilon_{2n}|\epsilon_{2n-1}||\epsilon_{2n-2}|\epsilon_{2n-3}| \\ &= d_{2n}d_{2n-1}d_{2n-2}d_{2n-3}, \\ d_{2n+2} &= K_{12}^{\frac{31}{65}}|\epsilon_{2n+2}| \\ &\le K_{12}^{\frac{31}{65}} K_{12}\epsilon_{2n+1}^{32} = \left(K_{12}^{\frac{3}{65}}\epsilon_{2n+1}\right)^{32} = d_{2n+1}^{32}. \end{aligned} \tag{26}$$

Evidently, from (26), we find

$$\begin{aligned} &d_1 \le d^4,\ d_2 \le d^{128},\ d_3 \le d^{134},\ d_4 \le d^{4288}, \\ &d_5 \le d^{4554},\ d_6 \le d^{145728},\ d_7 \le d^{154704}, \\ &d_8 \le d^{4950528},\ d_9 \le d^{5255514},\ d_{10} \le d^{168176448}. \end{aligned}$$

Our results concerning the order of convergence generated by (23) are summarized in the following theorem.

Theorem D. *Assume that the initial approximations x_0, x_{-1}, x_{-2}, x_{-3} are chosen so that $d_{-3} \le d$, $d_{-2} \le d$, $d_{-1} \le d < 1$ and $d_0 \le d < 1$. Then for the error of the sequences $\{x_{2n+1}\}_{n=0}^{\infty}$ and $\{x_{2n+2}\}_{n=0}^{\infty}$ determined by (23), we have*

$$\begin{aligned} d_{2n-1} &\le d^{\tau_{2n-1}}, \\ d_{2n} &\le d^{\tau_{2n}}, \end{aligned} \tag{27}$$

where

$$\tau_{m+4} = 33\tau_{m+2} + 33\tau_m, \quad m = 1, 2, \ldots \tag{28}$$

and the order of convergence of the iteration (23) is

$$\tau = \frac{\left(33+\sqrt{1221}\right)}{2}.$$

Proof. In our case, for the recursion

$$\tau_{m+4} = 33\tau_{m+2} + 33\tau_m,$$

characteristic polynomial is of the type

$$\rho^2 - 33\rho - 33 = 0. \tag{29}$$

Equation (29) has the roots:

$$\rho_1 = \frac{\left(33+\sqrt{1221}\right)}{2}, \quad \frac{\left(33-\sqrt{1221}\right)}{2}.$$

From the general iterative theory it follows that the order of convergence of the iteration procedure, defined by (23) is given by the only real root of equation (22) with magnitude greater than 1. On the other hand,

$$|\epsilon_{2n+1}| \le K_{12}^{-\frac{3}{65}} d_{2n+1}, \quad |\epsilon_{2n+2}| \le K_{12}^{-\frac{31}{65}} d_{2n+2},$$

and consequently we can conclude that the order of convergence of iteration (23) is

$$\tau = \frac{\left(33 + \sqrt{1221}\right)}{2} \approx 33.9714...$$

Thus, the theorem is proven.

III. Conclusion

If in the literature in terms of optimal Kung–Traub algorithm of order 64 appeared, we have shown that the acceleration in the light of our considerations is:

$$\tau = \frac{\left(65 + \sqrt{4485}\right)}{2} \approx 65.9851...$$

Intensively working scientific groups in this branch of numerical analysis should directed theirs efforts to make new interval numerical algorithms which are based on recently arised schemes which are optimal in the sense of Kung–Traub.

For methodical construction of numerical algorithms with result verification see Markov [17], [18] and his coauthors [4], [5], [19], [15].

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