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ORIGINAL ARTICLE

A new class of activation functions. Some related problems and applications

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***Abstract*—The cumulative distribution function (cdf) of the discrete two–parameter bathtub hazard distribution has important role in the fields of population dynamics, reliability analysis and life testing experiments. Also of interest to the specialists is the task of approximating the Heaviside function by new (cdf) in Hausdorff sense. We define new activation function and family of new recurrence generated functions and study the "saturation" by these families. In this paper we analyze some intrinsic properties of the new Topp–Leone–G–Family with baseline "deterministic–type" (cdf) – (NTLG–DT). Some numerical examples with real data from Biostatistics, Population dynamics and Signal theory, illustrating our results are given. It is shown that the study of the two characteristics - "confidential curves" and "super saturation" is a must when choosing the right model. Some related problems are discussed, as an example to the Approximation Theory.**

***Keywords*-two–parameter bathtub hazard distribution; "saturation" by: new activation function and family of new recurrence generated functions; Topp–Leone–G–Family with baseline "deterministic–type" (cdf) – (NTLG–DT); Heaviside function; Hausdorff distance; upper and lower bounds**

I. Introduction and Preliminaries

Definition 1. *Define the following deterministic (cdf) based on two–parameter bathtub hazard distribution [2]:*

$$M_\beta(t) = 1 - q^{e^{t^\beta}-1}, \qquad (1)$$

where $0 < q < 1$; $\beta > 0$, $t > 0$.

Definition 2. *The shifted Heaviside step function*

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is defined by

$$h_{t_0}(t) = \begin{cases} 0, & \text{if} \quad t < t_0, \\ [0,1], & \text{if} \quad t = t_0, \\ 1, & \text{if} \quad t > t_0 \end{cases} \quad (2)$$

Definition 3. *[3] The Hausdorff distance (the H–distance) $\rho(f,g)$ between two interval functions f,g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,*

$$\rho(f,g) = \max\{ \sup_{A \in F(f)} \inf_{B \in F(g)} ||A-B||,$$

$$\sup_{B \in F(g)} \inf_{A \in F(f)} ||A-B||\},$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t,x)|| = \max\{|t|,|x|\}$; hence the distance between the points $A=(t_A,x_A)$, $B=(t_B,x_B)$ in $\mathbb{R}^2$ is $||A-B|| = max(|t_A-t_B|,|x_A-x_B|)$.

Definition 4. *We define the following activation function:*

$$A(t;\beta) = \frac{q^{e^{-t^\beta}} - q^{e^{t^\beta}}}{q^{e^{-t^\beta}} + q^{e^{t^\beta}}}. \quad (3)$$

Definition 5. *Define the following family of new recurrence generated functions*

$$\begin{aligned} &A_{i+1}(t;\beta) = A_i(t + A_i(t;\beta);\beta), \\ &i = 0,1,2,\dots; \ A_0(t;\beta) = A(t;\beta). \end{aligned} \quad (4)$$

based on the function $A(t;\beta)$.

In [1] Bantan, Jamal, Chesneau and Elgarhy introduced a new power Topp–Leone–G–Family (NTL–G) of distribution with (cdf)

$$F(t) = e^{\alpha\beta\left(1-\frac{1}{G(t)}\right)} \left(2 - e^{\beta\left(1-\frac{1}{G(t)}\right)}\right)^\alpha \quad (5)$$

where $\alpha, \beta \in R^+$ and $G(t)$ is a (cdf) of a baseline continuous distribution.

The following result shows some inequalities involving $F(t)$ (see, Proposition 1 [1]):

$$e^{\alpha\beta\left(1-\frac{1}{G(t)}\right)} \left(2 - G(t)^\beta\right)^\alpha \le F(t) \le 2^\alpha e^{\alpha\beta\left(1-\frac{1}{G(t)}\right)}. \quad (6)$$

In this paper we study some properties of the new Topp–Leone–G–Family with baseline "deterministic–type" (cdf) – (NTLG–DT); $G(t) = 1 - q^{e^t - 1}$, where $0 < q < 1$.

Definition 6. *We define the following corresponding (cdf):*

$$Q(t) = e^{\alpha\beta\left(1-\frac{1}{1-q^{e^t-1}}\right)} \left(2 - e^{\beta\left(1-\frac{1}{1-q^{e^t-1}}\right)}\right)^\alpha \quad (7)$$

where $\alpha, \beta \in R^+$ and $0 < q < 1$.

II. Main Results

When studying the intrinsic properties of the family $M_\beta(t)$, it is also appropriate to study the "saturation" to the horizontal asymptote.

In this Section we give upper and lower estimates for the one–sided Hausdorff approximation of the Heaviside step–function $h_{t_0}(t)$ by means of family (1), where t_0 is the level of the "median".

A. The case $\beta = 1$.

Let t_0 is the unique positive root of the nonlinear equation $M_1(t_0) - \frac{1}{2} = 0$.

The one–sided Hausdorff distance d between $h_{t_0}(t)$ and the function (1) satisfies the relation

$$M_1(t_0 + d) = 1 - q^{e^{(t_0+d)}-1} = 1 - d. \quad (8)$$

The following theorem gives upper and lower bounds for d

Theorem 1. Let

$$\begin{aligned} &\beta = 1, \\ &q < \frac{2}{e^{2\left(\frac{e^{1.05}}{2.1}-1\right)}} \approx 0.971975. \end{aligned} \quad (9)$$

Then, for the one–sided Hausdorff distance d between $h_{t_0}(t)$ and the (cdf) – (1) the following inequalities hold:

$$d_l = \frac{1}{2.1(1+\frac{1}{2}\ln\frac{2}{q})} < d < \frac{\ln(2.1(1+\frac{1}{2}\ln\frac{2}{q}))}{2.1(1+\frac{1}{2}\ln\frac{2}{q})} = d_r. \quad (10)$$

Proof. In order to express d in terms of q, let us examine the function

$$f(d) = M_1(t_0 + d) - 1 + d.$$

From $f'(d) > 0$ we conclude that function $f(d)$ is strictly monotonically increasing.

Consider then the function

$$g(d) = -\frac{1}{2} + (1 + \frac{1}{2}\ln\frac{2}{q})d,$$

which approximates function f with $d \to 0$ as $O(d^2)$ (see, Fig. 1).

In addition $g'(d) > 0$.

We look for two reals d_l and d_r such that $g(d_l) < 0$ and $g(d_r) > 0$ (leading to $g(d_l) < d < g(d_r)$).

From (9) we have

$$g\left(d_l = \frac{1}{2.1(1 + \frac{1}{2}\ln\frac{2}{q})}\right) < 0,$$

$$g\left(d_r = \frac{\ln(2.1(1 + \frac{1}{2}\ln\frac{2}{q}))}{2.1(1 + \frac{1}{2}\ln\frac{2}{q})}\right) > 0$$

proving the estimates (10).

For example, for $\beta = 1$, $q = 0.1$ we have

$$d_l = 0.190639 < d = 0.230226 < 0.31596 = d_r$$

and for $\beta = 1$, $q = 0.9$ we have

$$d_l = 0.340317 < d = 0.355551 < 0.36682 = d_r.$$

B. The case $\beta \neq 1$.

For given $\beta \neq 1$ the one–sided Hausdorff distance d satisfies the relation

$$M_\beta(t_0 + d) = 1 - q^{e^{(t_0+d)^\beta} - 1} = 1 - d. \quad (11)$$

The reader may formulate the corresponding approximation problem following the ideas given in Theorem 1, and will be omitted.

We illustrate the "saturation" with the (cdf) – (1) for various β and fixed $q = 0.1$ (see, Fig. 2)

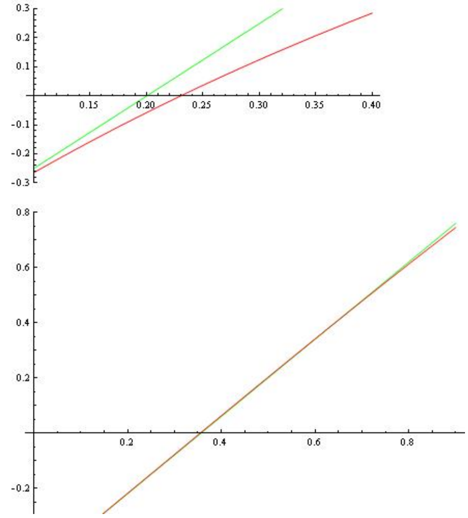

Fig. 1. The functions $f(d)$ and $g(d)$ for a) $\beta = 1$, $q = 0.1$; b) $\beta = 1$, $q = 0.9$.

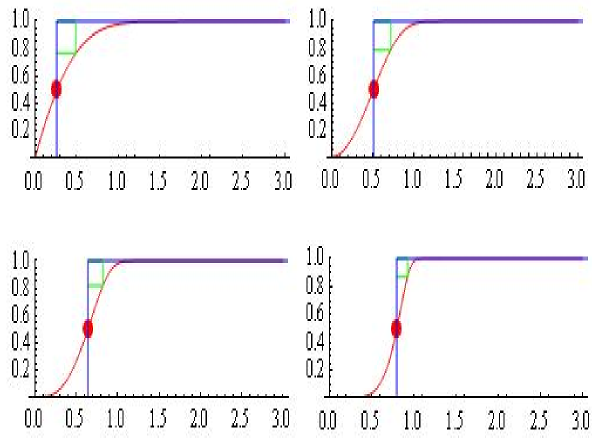

Fig. 2. a) $\beta = 1$, $q = 0.1$; $t_0 = 0.263156$; Hausdorff distance $d = 0.230226$; b) $\beta = 2$, $q = 0.1$; $t_0 = 0.512988$; Hausdorff distance $d = 0.208046$; c) $\beta = 3$, $q = 0.1$; $t_0 = 0.640823$; Hausdorff distance $d = 0.181048$; d) $\beta = 6$, $q = 0.1$; $t_0 = 0.800514$; Hausdorff distance $d = 0.127635$.

III. Some Applications.

It is well known that in many cases the existing modifications to the classical logistic and Gompertz models do not give very reliable results in approximating "specific data".

We examine the following "specific datasets":

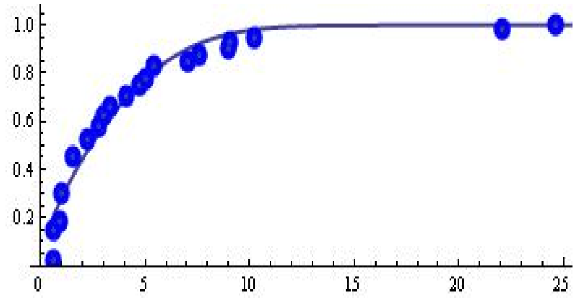

Fig. 3. The fitted model (1).

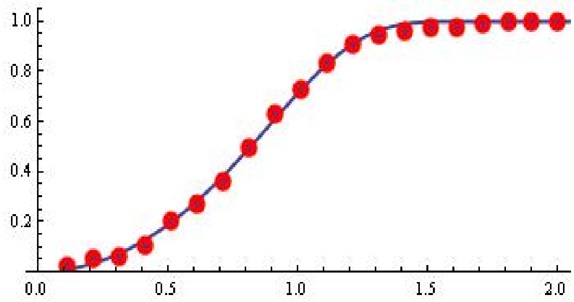

Fig. 4. The fitted model (1).

Example 1. We analyze the following data [4]

$data\_Communication := \{\{0.584, 0.027\},$
$\{0.649, 0.147\}, \{0.909, 0.187\}, \{1.039, 0.303\},$
$\{1.558, 0.453\}, \{2.208, 0.527\}, \{2.792, 0.580\},$
$\{3.052, 0.627\}, \{3.312, 0.657\}, \{4.091, 0.707\},$
$\{4.740, 0.753\}, \{5, 0.780\}, \{5.390, 0.827\},$
$\{7.078, 0.853\}, \{7.597, 0.877\}, \{8.961, 0.903\},$
$\{9.091, 0.927\}, \{10.195, 0.950\}, \{22.078, 0.980\},$
$\{24.610, 1\}\};$

The cdf $M_\beta(t)$ for $\beta = 0.484411$ and $q = 0.82547$ is visualized on Fig. 3.

Example 2. Analysis of "$data\_Nicotine$" [5]

$data\_Nicotine :=$
$\{\{0.11, 0.021\}, \{0.21, 0.053\}, \{0.31, 0.063\},$
$\{0.41, 0.105\}, \{0.51, 0.2\}, \{0.61, 0.274\},$
$\{0.71, 0.358\}, \{0.81, 0.495\}, \{0.91, 0.632\},$
$\{1.01, 0.726\}, \{1.11, 0.832\}, \{1.21, 0.905\},$
$\{1.31, 0.942\}, \{1.41, 0.958\}, \{1.51, 0.974\},$
$\{1.61, 0.979\}, \{1.71, 0.989\}, \{1.81, 1\},$
$\{1.9, 1\}, \{2, 1\}\};$

After that using the model $M_\beta(t)$ for $\beta = 1.98567$ and $q = 0.485475$ we obtain the fitted model (see, Fig. 4).

Example 3. Analysis of data "Biomass produced by *Paesilomyces lilacinus 6029*" [6].

After that using the model $M^*_\beta(t) = \omega M_\beta(t)$ for $\omega = 10.521$, $\beta = 0.805824$ and $q = 0.97915$ we obtain the fitted model (see, Fig. 5).

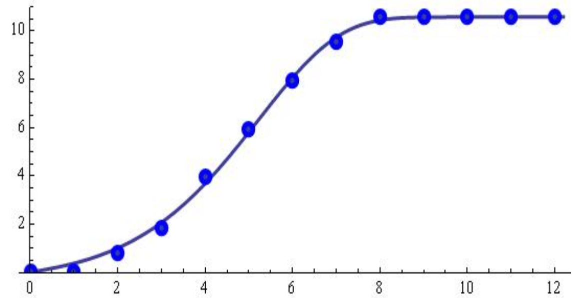

Fig. 5. The fitted model.

The new activation function.

We define the following activation function:

$$A(t;\beta) = \frac{q^{e^{-t^\beta}} - q^{e^{t^\beta}}}{q^{e^{-t^\beta}} + q^{e^{t^\beta}}}. \qquad (12)$$

In antenna-feeder technique most often occurred signals are of types shown on Fig. 6 – Fig. 7.

For β even, the corresponding approximation using model (7) is shown in Fig. 6.

For β odd, the corresponding approximation using new activation function $A(t;\beta)$ is shown in Fig. 7.

A family of recurrence generated functions based on the $A(t;\beta)$.

Let us consider the following family of recurrence generated functions

$$\begin{aligned} &A_{i+1}(t;\beta) = A_i(t + A_i(t;\beta);\beta), \\ &i = 0, 1, 2, \dots;\ \ A_0(t;\beta) = A(t;\beta), \end{aligned} \qquad (13)$$

based on the function $A(t;\beta)$.

Let for instance $\beta = 1$.

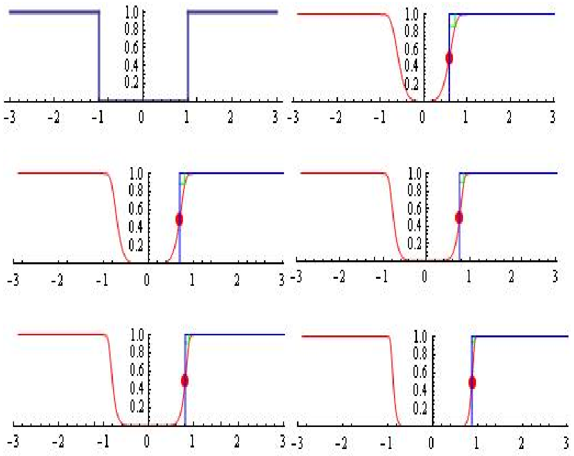

Fig. 6. The function $A(t;\beta)$; $\beta = 4, q = 0.01, t_0 = 0.587335$; Hausdorff distance $d = 0.138899$; $\beta = 6, q = 0.01, t_0 = 0.701333$; Hausdorff distance $d = 0.111603$; $\beta = 8, q = 0.01, t_0 = 0.766378$; Hausdorff distance $d = 0.0992629$; $\beta = 10, q = 0.01, t_0 = 0.808266$; Hausdorff distance $d = 0.0867535$; $\beta = 16, q = 0.01, t_0 = 0.87543$; Hausdorff distance $d = 0.0632673$.

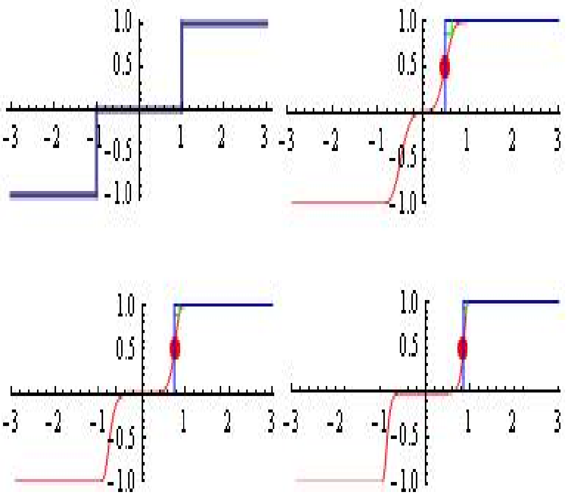

Fig. 7. The function $A(t;\beta)$; $\beta = 3, q = 0.01, t_0 = 0.491867$; Hausdorff distance $d = 0.152538$; $\beta = 7, q = 0.01, t_0 = 0.737794$; Hausdorff distance $d = 0.107003$; $\beta = 13, q = 0.01, t_0 = 0.848962$; Hausdorff distance $d = 0.073086$.

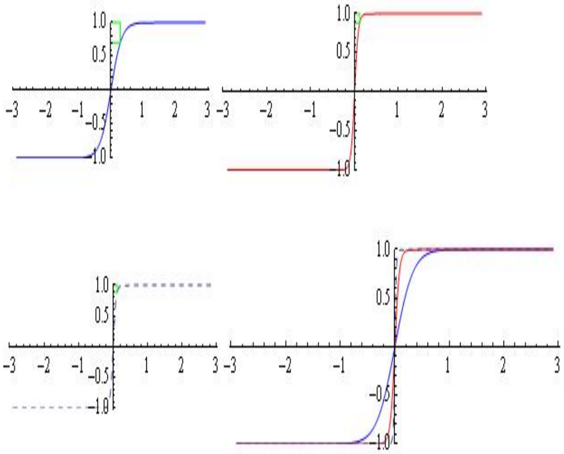

Fig. 8. The recurrence generated family: $A_0(t)$ (blue), $A_1(t)$ (red) and $A_2(t)$ (dashed).

The recurrence generated family: $A_0(t), A_1(t)$ and $A_2(t)$ is visualized on Fig. 8.

Some properties of the new Topp–Leone–G–Family with baseline "deterministic–type" (cdf) – (NTLG–DT) $Q(t)(7)$.

We study the Hausdorff approximation of the Heaviside step function $h_{t_0}(t)$ where t_0 is the "median" by families of the new Topp–Leone–G–Family with baseline "deterministic–type" (cdf) – (NTLG–DT).

The obtained two-sides estimations (see Proposition 1. [1]) in particular case with usage of the baseline "deterministic–type" (cdf) for $\alpha = 0.9;\ \beta = 0.3;\ q = 0.1$

$$e^{\alpha\beta\left(1-\frac{1}{1-q^{e^t-1}}\right)}\left(2-\left(1-q^{e^t-1}\right)^{\beta}\right)^{\alpha} \leq Q(t) \leq 2^{\alpha} e^{\alpha\beta\left(1-\frac{1}{1-q^{e^t-1}}\right)} \tag{14}$$

are given in Fig. 9 a.

Let t_0 is the value for which $Q(t_0) = \frac{1}{2}$.

The Hausdorff distance d between the function $h_{t_0}(t)$ and $Q(t)$ satisfies the relation

$$Q(t_0 + d) = 1 - d. \tag{15}$$

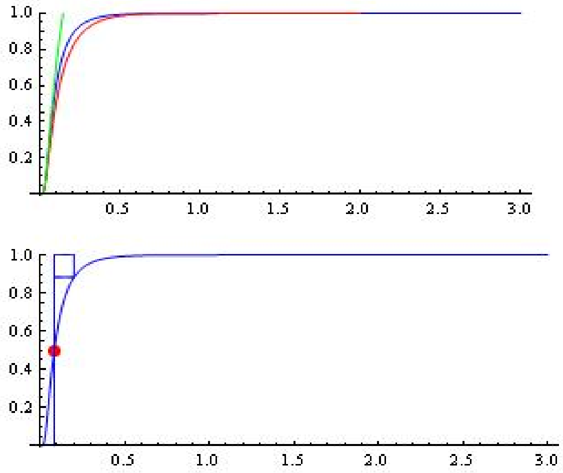

Fig. 9. a) The two-sides estimations (14) for $\alpha = 0.9$; $\beta = 0.3$; $q = 0.1$; b) The model $Q(t)$ for $\alpha = 0.9$; $\beta = 0.3$; $q = 0.1$, $t_0 = 0.0852097$; H–distance $d = 0.116811$

For fixed $\alpha = 0.9$; $\beta = 0.3$; $q = 0.1$ we find $t_0 = 0.0852097$ and from the nonlinear equation (15) we have $d = 0.116811$ (see, Fig. 9 b).

From Fig. 9 it can be seen that these estimations can be used as "confidence bounds", which are extremely useful for the specialists in the choice of model for cumulative data approximating in areas of Biostatistics, Population dynamics, Growth theory, Debugging and Test theory, Computer viruses propagation, Financial and Insurance mathematics.

For other results, see [8]–[53], [59].

IV. Concluding remarks.

The results obtained in this article can be successfully continued.

1. For example, we study the Hausdorff approximation of the Heaviside step function $h_{t_0}(t)$ where t_0 is the "median" by families of the new Topp–Leone–G–Family $Q_1(t)$ with baseline "deterministic–inverse–type" (cdf) – (NTLG–DIT) $G(t) = q^{e^{\frac{1}{t}}-1}$, where $0 < q < 1$,

$$Q_1(t) = e^{\alpha\beta\left(1-\frac{1}{q^{e^{\frac{1}{t}}-1}}\right)}\left(2 - e^{\beta\left(1-\frac{1}{q^{e^{\frac{1}{t}}-1}}\right)}\right)^{\alpha} \tag{16}$$

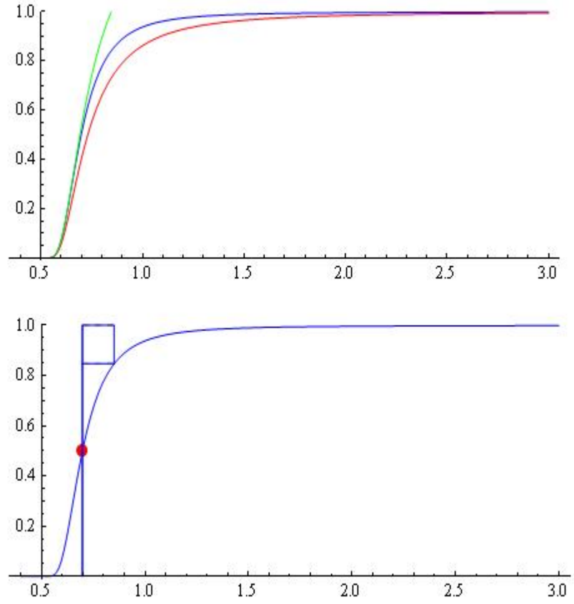

Fig. 10. a) The two-sided bounds (17) for $\alpha = 0.6$; $\beta = 0.1$; $q = 0.4$; b) The model $Q_1(t)$ for $\alpha = 0.6$; $\beta = 0.1$; $q = 0.4$, $t_0 = 0.698075$; H–distance $d = 0.153113$

The obtained two-sided bounds (see Proposition 1. [1]) in particular case with usage of the baseline "deterministic–inverse–type" (cdf) for $\alpha = 0.6,\ \beta = 0.1,\ q = 0.4$,

$$e^{\alpha\beta\left(1-\frac{1}{q^{e^{\frac{1}{t}}-1}}\right)}\left(2 - q^{\beta\left(e^{\frac{1}{t}}-1\right)}\right)^{\alpha} \tag{17}$$

$$\le Q_1(t) \le 2^{\alpha} e^{\alpha\beta\left(1-\frac{1}{q^{e^{\frac{1}{t}}-1}}\right)}$$

are given in Fig. 10 a.

Example 4. Storm worm one of the most biggest cyber threats of 2008.

We analyze the following data [7]

$$data\_Storm\_IDs := \{\{1, 0.843\}, \{4, 0.926\}, \{5, 0.954\}, \{6, 0.967\}, \{7, 0.976\}, \{8, 0.981\}, \{9, 0.985\}, \{10, 0.991\}, \{22, 0.995\}, \{38, 0.997\}, \{51, 0.998\}, \{64, 0.9985\}, \{74, 0.999\}, \{83, 1\}, \{100, 1\}, \{367, 1\}\}$$

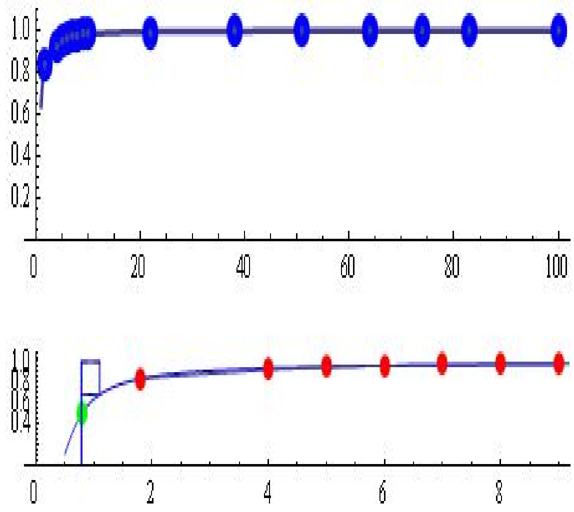

Fig. 11. The fitted model $Q_1(t)$.

The cdf $Q_1(t)$ for $\alpha = 0.14146$, $\beta = 161.078891$, $q = 0.9$ is visualized on Fig. 11.

Exploring both features - "confidential curves" and "super saturation" is a must when choosing the right model.

2. Following the ideas given in [54]–[56] we consider the following new differential model:

$$\begin{cases} \frac{dy(t)}{dt} = ky(t)s(t) = ky(t)q^{e^t - 1} \\ y(t_0) = y_0 \end{cases} \quad (18)$$

where $k > 0$ and $0 < q < 1$.

The general solution of the differential equation (18) is of the following form:

$$y(t) = y_0 e^{\frac{k}{q} Ei(e^t \ln q) - \frac{k}{q} Ei(\ln q)} \quad (19)$$

where $Ei(.)$ is the traditional exponential integral.

The new "growth" function $y(t)$ and the "input function" $s(t) = q^{e^t - 1}$ are visualized on Fig. 12–Fig. 13.

Example 4. We will analyze a sample of experimental data obtained by the biologist T. Carlson in 1913 about the development of Saccharomyces culture in nutrient medium (see, for example [58], [57]).

After that using the model $M^*(t) = \omega e^{\frac{k}{q} Ei(e^t \ln q) - \frac{k}{q} Ei(\ln q)}$ for $k = 0.293574$,

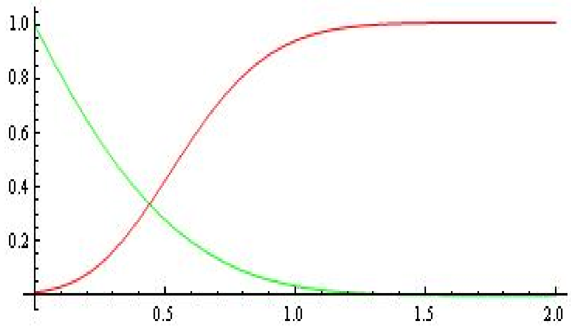

Fig. 12. The "growth" function $y(t)$–(red) and $s(t)$–(green) for $k = 12.6$; $q = 0.14$; $y_0 = 0.01$.

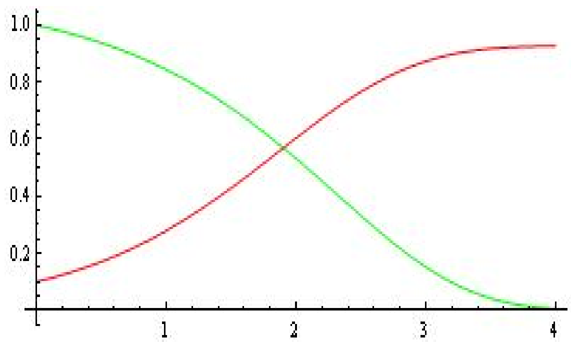

Fig. 13. The "growth" function $y(t)$–(red) and $s(t)$–(green) for $k = 1.1$; $q = 0.906$; $y_0 = 0.1$.

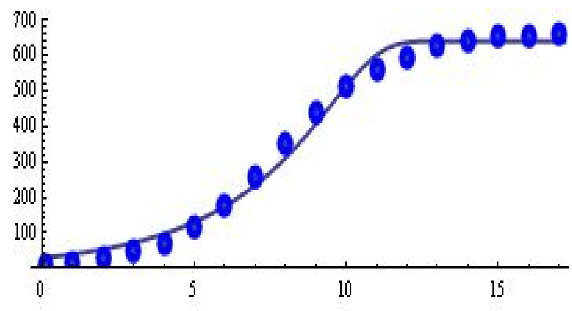

Fig. 14. The fitted model $M^*(t)$.

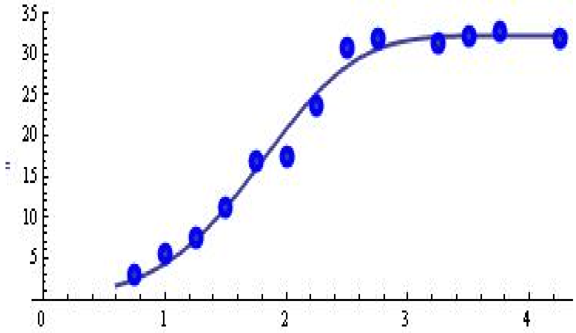

Fig. 15. The fitted model $M^*(t)$.

$q = 0.999983$ and $\omega = 30.114$ we obtain the fitted model (see, Fig. 14).

Example 5. Analysis of data "Biomass produced by *Paesilomyces sinclairi ascomycota*".

After that using the model $M^*(t)$ for $\omega = 0.305247$, $k = 3.01914$ and $q = 0.83$ we obtain the fitted model (see, Fig. 15).

The general solution $y(t)$ has been applied widely in life testing experiments and debugging theory.

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