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SVLIAR age-of-infection and -immunity structured epidemic model of COVID-19 dynamics

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Abstract: **Stability analysis of nonlinear age-of-infection and -immunity structured SVLIAR-type model of susceptible, vaccinated, latent, COVID-19 infected, asymptomatic and recovered sub-classes of population dynamics is carried out in this paper. The SVLIAR model uses five age variables – age of vaccine immunity of vaccinated individuals, age of virus infection in organism during incubation period of latent individuals, "age" of infectious disease treatment of infected individuals, age of asymptomatic infectious dis-ease of asymptomatic individuals, "age" of immunity of organism after recovering of recovered individuals.**

Individuals can move from one subclass to another when these age variables take some fixed values, that is the processes in sub-classes are adjusted and synchronized by age variables. The conditions for the existence of disease-free and unique endemic equilibria and their local asymptotic stability were obtained.

The local asymptotic stability/instability of endemic equilibrium of SVLIAR model is defined by criterion, which relates the demographic characteristics of population, infection disease characteristics (disease-induced death rate, death rate induced by the complications after disease), characteristics of vaccination (fraction of fully vaccinated susceptibles per unit of time, vaccination efficacy) and characteristics of age variables (their maximum values) of sub-classes.

These theoretical results help understand better the conditions of transmission dynamics of the COVID-19 induced disease.

Keywords: **age-structured model, vaccination, reinfection, adjustment, COVID-19**

I. Introduction

This study is focused on a qualitative analysis of transmission dynamics of the COVID-19 induced disease in sub-classes of susceptible, vaccinated, latent, infected, asymptomatic and recovered individuals of population. The mostly used methods of theoretical analysis of transmission dynamics of infectious diseases in epidemic models are based on the age-structured models of population dynamics [1–8] which relate the age-dependent demographic parameters of susceptible, infected, and recovered sub-classes of population with characteristics of infection-induced disease transmission. Be-cause such models use pretty complex and accurate mathematical methods for simulation, they help us understand better the features of mechanisms, risks, dynamics and mitigation of pandemic diseases.

However, the characteristic time scale of many infectious diseases (including COVID-19) is about several months that is significantly less than the characteristic time scale of demographic processes of population – several dozen years. That is why such class of epidemic age-structured models neglects the age dependent modeling of demo-graphic processes and considers the unstructured equation of susceptible subclass dynamics and at the same time the age-of-infection structured equation of infected subclass dynamics [9–11]. The new independent variable – age-of-infection can play a role of parameter of accurate adjustment of dynamical processes in susceptible and infected sub-classes.

In works [12, 13] authors introduce successfully the age-of-infection variable in age-structured equations to describe disease progression through multiple infectious stages (as in the case of HIV, hepatitis B and hepatitis C) [12] and for partition of sub-classes of individuals infected with acute HBV and chronic HBV carriers in hepatitis B transmission model [13].

Another important aspect of studying the infectious disease transmission is the modelling of vaccination of population. Vaccination of susceptible subclass of population plays a crucial role for disease mitigation and decreasing of disease activity in practice, including COVID-19 disease. Such problems are studied theoretically in works [14, 15] for epidemic model of SVLIAR-type based on unstructured equations of susceptible, vaccinated, latent, infected, asymptomatic and recovered individuals of population. Authors analyze the effect of vaccination in an SVLIAR model with demography by adding a compartment for vaccinated individuals and considering disease-induced death, imperfect and waning vaccination protection as well as waning infections-acquired immunity.

The new age-of-infection and -immunity structured SVLIAR-type model studied in this paper is based on the structured equations of sub-classes dynamics with introduced 5 in-dependent variables for each subclass (except subclass of susceptible individuals): a_1 – age of vaccine immunity of vaccinated individuals, a_2 – age of virus infection in organism during incubation period of latent individuals, a_3 – age (time period) of treatment of infectious disease of infected individuals, a_4 – age of asymptomatic infectious disease of asymptomatic individuals, a_5 – age (time period) of immunity of organism after recovering of recovered individuals. Each variable a_i runs the internal sub-process that is specific for each particular subclass and allows us to adjust and synchronize all sub-processes with each other. Individuals can move from one sub-class to another when these age variables take the maximum values, that is the processes in sub-classes are adjusted and synchronized by age variables. Such detailed age-structured epidemic model with adjustment of sub-processes provides the more accurate simulation of transmission dynamics of infectious disease and can be a basis for further theoretical analysis and simulation of different aspects of COVID-19 epidemic in more complex models.

Local asymptotic stability/instability of disease-free and endemic equilibria of age-of-infection and -immunity structured system is defined in the paper by new derived criteria which relate the maximum ages of each variable (maximum values of a_i) with demographic characteristics of population (birth and death rates), disease-induced death rate, death rate induced by the complications after COVID-19 disease, fraction of fully vaccinated susceptibles per unit of time, vaccination efficacy, rate of disease transmission and the other characteristics of the model.

Age-of-infection and -immunity structured SVLIAR epidemic model is considered in Section 2. Existence of disease-free equilibrium and its local asymptotic stability are studied in terms of the basic reproduction number in Section 3. Existence of endemic equilibrium and its local asymptotic stability [7, 16–18] are studied in terms of the basic reproduction number. Conditions of existence of a unique endemic equilibrium are derived in Sections 4, 5. This means, among other things, that additional compartment for vaccinated individuals has no effects on increasing of number of endemic equilibria but effects on its stability. Several concluding remarks are given in Section 6. The existence theorem, explicit recurrent formula for the solution of the age-structured SVLIAR model and numerical method with simulations (like in works [1, 2]) are beyond the aim and scope of this paper due to the complexity of the model and will be the subject of our further study.

II. The model

Age-of-infection and -immunity structured SVLIAR epidemic model considers transmission dynamics of the COVID-19 virus disease in population which consists of the following sub-classes (Figure 1):

- susceptible (non-infected),
- vaccinated (people are immune after 1st and 2nd vaccinations),
- latent (infected individuals without disease symptoms when virus develops within its incubation period),
- infected (ill individuals with explicit symptoms),
- asymptomatic (individuals with symptoms free form of infectious disease),
- recovered (people are immune after infectious disease).

The quantity of susceptible individuals is described by $S(t)$. The age-specific density of vaccinated sub-classes of individuals is $V(a_1, t)$, $a_1 \in [0, a_d^{(1)}]$, $t \geq 0$, where $a_d^{(1)}$ is a period of time when vaccination-induced immunity starts to wain (maximum age of vaccination-induced immunity). The total number of vaccinated individuals is

$$N_V(t) = \int_0^{a_d^{(1)}} V(a_1,t)da_1.$$

The age-specific density of latent individuals is $L(a_2,t)$, $a_2 \in [0, a_d^{(2)}]$, $t \geq 0$, where $a_d^{(2)}$ is a maximum incubation period of virus infection in organ-ism. The number of all latent individuals is

$$N_L(t) = \int_0^{a_d^{(2)}} L(a_2,t)da_2.$$

The age-specific density of infected individuals, which have the symptoms of disease, is $I(a_3,t)$, $a_3 \in [0, a_d^{(3)}]$, $t \geq 0$, where $a_d^{(3)}$ is a maximum period of infectious disease (or disease treatment). The number of infected individuals is

$$N_I(t) = \int_0^{a_d^{(3)}} I(a_3,t)da_3.$$

The age-specific density of asymptomatic individuals, which are infected, sick and do not have the symptoms of disease, is $A(a_4,t)$, $a_4 \in [0, a_d^{(4)}]$, $t \geq 0$, where $a_d^{(4)}$ is a maximum period of asymptomatic infectious disease. The number of asymptomatic individuals is

$$N_A(t) = \int_0^{a_d^{(4)}} A(a_4,t)da_4.$$

The age-specific density of recovered individuals is $R(a_5,t)$, $a_5 \in [0, a_d^{(5)}]$, $t \geq 0$, where $a_d^{(5)}$ is a period when disease-induced immunity of individuals starts to wain after recovering (maximum age of disease-induced immunity). The number of recovered individuals is defined as

$$N_R(t) = \int_0^{a_d^{(5)}} R(a_5,t)da_5.$$

We will assume further that $a_d^{(2)} + a_d^{(5)} > a_d^{(1)}$, that is the sum of infection incubation period and period of disease-induced immunity waning in recovered individuals is bigger than period of vaccination-induced immunity waning in vaccinated individuals. The series of buster vaccinations and cases with new COVID-mutations for which the current vaccine is not efficient are not considered in this study.

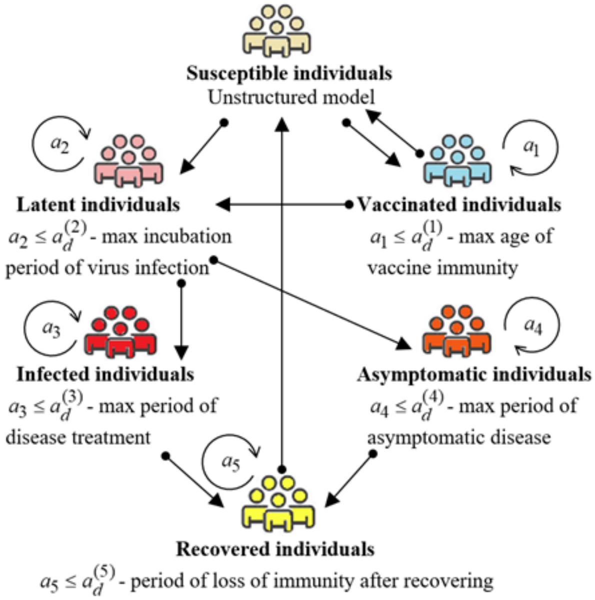

Fig. 1: Schematic structure of SVLIAR model.

We arrive to the autonomous SVLIAR age-structured epidemic model

$$\frac{dS(t)}{dt} = -(\mu - b + q + f(t))S(t) + V(a_d^{(1)},t) + R(a_d^{(5)},t), \tag{1}$$

$$\frac{\partial V(a_1,t)}{\partial t} + \frac{\partial V(a_1,t)}{\partial a_1} = -(\mu + (1-\sigma)f(t))V(a_1,t), \tag{2}$$

$$\frac{\partial L(a_2,t)}{\partial t} + \frac{\partial L(a_2,t)}{\partial a_2} = -\mu L(a_2,t), \tag{3}$$

$$\frac{\partial I(a_3,t)}{\partial t} + \frac{\partial I(a_3,t)}{\partial a_3} = -(\mu + \gamma)I(a_3,t), \tag{4}$$

$$\frac{\partial A(a_4,t)}{\partial t} + \frac{\partial A(a_4,t)}{\partial a_4} = -\mu A(a_4,t), \tag{5}$$

$$\frac{\partial R(a_5,t)}{\partial t} + \frac{\partial R(a_5,t)}{\partial a_5} = -(\mu + v)R(a_5,t), \tag{6}$$

where μ is a natural death rate, b is a birth rate, γ is a disease-induced death rate, v is a death rate induced by the complications after disease, q is a fraction of fully vaccinated susceptibles per unit of time, σ is a vaccination efficacy. Vaccinated individuals become susceptibles when they lost immunity after vaccina-tion at age of vaccination $a_d^{(1)}$. Recovered individuals become susceptibles when they lost immunity after full treatment at the maximum age of after-disease immunity $a_d^{(5)}$. The force of infection is defined as:

$$f(t) = \beta(N_I(t) + \eta N_A(t)), \tag{7}$$

where $\beta > 0$ is a rate of transmission, $\eta > 0$ is a modification of transmission for asymptomatic [14]. Eqs. (1)–(6) are completed by the non-negative initial values:

$$\begin{aligned} S(0) &= S_0, & V(a_1,0) &= 0, \\ L(a_2,0) &= L_0(a_2), & I(a_3,0) &= I_0(a_3), \\ A(a_4,0) &= 0, & R(a_5,0) &= 0. \end{aligned} \tag{8}$$

Density of newly vaccinated individuals $V(0,t)$ is the sum of the number of vaccinated arrivals and susceptibles per unit of time:

$$V(0,t) = qS(t). \tag{9}$$

Density of new latent individuals $L(0,t)$ is defined through the sum of the number of infected susceptibles and infected vaccinated individuals with low immunity due to the weak efficacy of vaccine:

$$L(0,t) = f(t)(S(t) + (1-\sigma)N_V(t)). \tag{10}$$

Density of just infected individuals $I(0,t)$ is a $(1-\rho)$–fraction of a density of latent individuals which have the symptoms of disease after the incubation period of infection:

$$I(0,t) = (1-\rho)L(a_d^{(2)},t). \tag{11}$$

Density of new asymptomatic individuals $A(0,t)$ is a ρ-fraction of density of latent individuals which do not have the symptoms of disease after the incubation period of infection:

$$A(0,t) = \rho L(a_d^{(2)},t). \tag{12}$$

Density of new recovered individuals $R(0,t)$ is the sum of densities of fully treated infected individuals and recovered asymptomatic individuals:

$$R(0,t) = I(a_d^{(3)},t) + A(a_d^{(4)},t). \tag{13}$$

When system (1)–(13) degenerates to the system of nonlinear ODE, it becomes:

$$\begin{aligned} \dot{S}(t) &= -(\mu - b + q + f(t))S(t) + V(a_d^{(1)},t) \\ &\quad + R(a_d^{(5)},t), \end{aligned} \tag{14}$$

$$\begin{aligned} \dot{N}_V(t) &= -(\mu + (1-\sigma)f(t))N_V(t) + qS(t) \\ &\quad - V(a_d^{(1)},t), \end{aligned} \tag{15}$$

$$\begin{aligned} \dot{N}_L(t) &= -\mu N_L(t) + f(t)(S(t) + (1-\sigma)N_V(t)) \\ &\quad - L(a_d^{(2)},t), \end{aligned} \tag{16}$$

$$\begin{aligned} \dot{N}_I(t) &= -(\mu+\gamma)N_I(t) + (1-\rho)L(a_d^{(2)},t) \\ &\quad - I(a_d^{(3)},t), \end{aligned} \tag{17}$$

$$\begin{aligned} \dot{N}_A(t) &= -\mu N_A(t) + \rho L(a_d^{(2)},t) \\ &\quad - A(a_d^{(4)},t), \end{aligned} \tag{18}$$

$$\begin{aligned} \dot{N}_R(t) &= -(\mu+v)N_R(t) + I(a_d^{(3)},t) + A(a_d^{(4)},t) \\ &\quad - R(a_d^{(5)},t). \end{aligned} \tag{19}$$

Summarizing Eqs. (14)–(19) yields the balance equation for the system (1)–(13):

$$\begin{aligned} &\dot{S}(t) + \dot{N}_V(t) + \dot{N}_L(t) + \dot{N}_I(t) + \dot{N}_A(t) + \dot{N}_R(t) \\ &\quad = bS(t) - \gamma N_I(t) - vN_R(t) - \mu\big(S(t) + N_V(t) \\ &\quad + N_L(t) + N_I(t) + N_A(t) + N_R(t)\big). \end{aligned} \tag{20}$$

Thus, the change in total population size over the given time period is due to the difference between newborn individuals and those who died during COVID-19 illness, died from complication after suffering COVID-19 illness and died of natural causes (or other, non-COVID-19 disease) over the given time period.

III. Trivial and disease-free equilibria

It is easy to verify that trivial equilibrium of the system (1)–(13) always exists. The disease-free equilibrium (DFE):

$$\begin{aligned} &S_0^* > 0,\ V_0^*(a_1) \geq 0,\ L_0^*(a_2) = 0, \\ &I_0^*(a_3) = 0,\ A_0^*(a_4) = 0,\ R_0^*(a_5) = 0, \\ &N_{L0}^* = 0,\ N_{I0}^* = 0,\ N_{A0}^* = 0,\ N_{R0}^* = 0, \\ &N_{V0}^* = \int_0^{a_d^{(1)}} V_0^*(a_1)da_1 > 0, \end{aligned}$$

satisfies the system:

$$0 = -(\mu - b + q)S_0^* + V_0^*(a_d^{(1)}), \tag{21}$$

$$\frac{dV_0^*(a_1)}{da_1} = -\mu V_0^*(a_1), \tag{22}$$

$$V_0^*(0) = qS^*, \tag{23}$$

which has a solution:

$$\begin{aligned} V_0^*(a_1) &= qS_0^* \exp(-\mu a_1), \\ V_0^*(a_d^{(1)}) &= S_0^*(\mu - b + q). \end{aligned} \tag{24}$$

Plugging $V_0^*(a_d^{(1)})$ from the second equation into the first one yields:

$$S_0^*\big(\mu - b + q(1 - \exp(-\mu a_d^{(1)}))\big) = 0.$$

Thus, we arrive at Statement 1.

Statement 1. *If coefficients of stationary system (21)–(23) satisfy condition*

$$R_0 = \frac{b}{\mu + q\Big(1-\exp\big(-\mu a_d^{(1)}\big)\Big)} = 1 \tag{25}$$

there exists the disease-free equilibrium of the system (1)–(13):

$$\begin{aligned}&S_0^* = S_0,\ V_0^*(a_1) = qS_0\exp\big(-\mu a_1\big),\ L_0^*(a_2)=0,\\ &I_0^*(a_3)=0,\ A_0^*(a_4)=0,\ R_0^*(a_5)=0,\\ &N_{L0}^*=0,\ N_{I0}^*=0,\ N_{A0}^*=0,\ N_{R0}^*=0,\\ &N_{V0}^* = \frac{qS_0}{\mu}\Big(1-\exp\big(-\mu a_d^{(1)}\big)\Big).\end{aligned} \tag{26}$$

It is easy to verify that DFE (26) is a particular stationary solution of system (1)–(13) with initial values $L_0(a_2)=0$, $I_0(a_3)=0$.

Condition $R_0 < 1$ holds if birth rate is relatively small $b < \mu + q\big(1-\exp(-\mu a_d^{(1)})\big)$, that is susceptible subclass is extinguishing and system has only trivial equilibrium. While condition $R_0 > 1$ holds if birth rate is relatively large $b > \mu + q\big(1-\exp(-\mu a_d^{(1)})\big)$, that is susceptible subclass is a growing population.

Linearizing system (1)–(6) at the DFE:

$$\begin{aligned}&S_0^* = S_0 > 0,\ V_0^*(a_1)\ge 0,\ N_{V0}^* > 0,\\ &L_0^*(a_2)=0,\ N_{L0}^*=0,\ I_0^*(a_3)=0,\ N_{I0}^*=0,\\ &A_0^*(a_4)=0,\ N_{A0}^*=0,\ R_0^*(a_5)=0,\ N_{R0}^*=0,\end{aligned}$$

we arrive at the system for perturbations:

- for S_0^*:
$$\bar{\xi}_{s0}(t) = \xi_{s0}\exp(\lambda t)$$
- for $V_0^*(a_1)$ and N_{V0}^*:
$$\bar{\psi}_{v0}(a_1,t) = \psi_{v0}(a_1)\exp(\lambda t)$$
$$\bar{\xi}_{v0}(t) = \int_0^{a_d^{(1)}} \bar{\psi}_{v0}(a_1,t)da_1 = \xi_{v0}\exp(\lambda t)$$
- for L_0^* and N_{L0}^*:
$$\bar{\psi}_{l0}(a_2,t) = \psi_{l0}(a_2)\exp(\lambda t)$$
$$\bar{\xi}_{l0}(t) = \int_0^{a_d^{(2)}} \bar{\psi}_{l0}(a_2,t)da_2 = \xi_{l0}\exp(\lambda t)$$
- for I_0^* and N_{I0}^*:
$$\bar{\psi}_{i0}(a_3,t) = \psi_{i0}(a_3)\exp(\lambda t)$$
$$\bar{\xi}_{i0}(t) = \int_0^{a_d^{(3)}} \bar{\psi}_{i0}(a_3,t)da_3 = \xi_{i0}\exp(\lambda t)$$
- for A_0^* and N_{A0}^*:
$$\bar{\psi}_{a0}(a_4,t) = \psi_{a0}(a_4)\exp(\lambda t)$$
$$\bar{\xi}_{a0}(t) = \int_0^{a_d^{(4)}} \bar{\psi}_{a0}(a_4,t)da_4 = \xi_{a0}\exp(\lambda t)$$
- for R_0^* and N_{R0}^*:
$$\bar{\psi}_{r0}(a_5,t) = \psi_{r0}(a_5)\exp(\lambda t)$$
$$\bar{\xi}_{r0}(t) = \int_0^{a_d^{(5)}} \psi_{r0}(a_5,t)da_5 = \xi_{r0}\exp(\lambda t)$$

Then:

$$\begin{aligned}0 = &-(\lambda+\mu-b+q)\xi_{s0} - \beta(\xi_{i0}+\eta\xi_{a0})S_0\\ &+\psi_{v0}(a_d^{(1)}) + \psi_{r0}(a_d^{(5)}),\end{aligned} \tag{27}$$

$$\begin{aligned}\frac{d\psi_{v0}(a_1)}{da_1} &= -(1-\sigma)\beta(\xi_{i0}+\eta\xi_{a0})V_0^*(a_1)\\ &\quad -(\lambda+\mu)\psi_{v0}(a_1)\end{aligned} \tag{28}$$

$$\frac{d\psi_{l0}(a_2)}{da_2} = -(\lambda+\mu)\psi_{l0}(a_2) \tag{29}$$

$$\frac{d\psi_{i0}(a_3)}{da_3} = -(\lambda+\mu+\gamma)\psi_{i0}(a_3) \tag{30}$$

$$\frac{d\psi_{a0}(a_4)}{da_4} = -(\lambda+\mu)\psi_{a0}(a_4) \tag{31}$$

$$\frac{d\psi_{r0}(a_5)}{da_5} = -(\lambda+\mu+v)\psi_{r0}(a_5) \tag{32}$$

Eqs. (27)–(32) are completed by the following boundary conditions:

$$\psi_{v0}(0) = q\xi_{s0} \tag{33}$$

$$\psi_{l0}(0) = \beta(S_0+(1-\sigma)N_{V0}^*)(\xi_{i0}+\eta\xi_{a0}) \tag{34}$$

$$\psi_{i0}(0) = (1-\rho)\psi_{l0}(a_d^{(2)}) \tag{35}$$

$$\psi_{a0}(0) = \rho\psi_{l0}(a_d^{(2)}) \tag{36}$$

$$\psi_{r0}(0) = \psi_{i0}(a_d^{(3)}) + \psi_{a0}(a_d^{(4)}) \tag{37}$$

Characteristic equation of stationary system (27)–(37) derived in Appendix A is:

$$\begin{aligned}R_1(\lambda) = &\beta S_0\bigg(1+\frac{(1-\sigma)q}{\mu}\Big(1-\exp\big(-\mu a_d^{(1)}\big)\Big)\bigg)\\ &\times\exp\big(-(\lambda+\mu)a_d^{(2)}\big)\Bigg(\rho\eta\frac{1-\exp\big(-(\lambda+\mu)a_d^{(4)}\big)}{\lambda+\mu}\\ &+(1-\rho)\frac{1-\exp\big(-(\lambda+\mu+\gamma)a_d^{(3)}\big)}{\lambda+\mu+\gamma}\Bigg) = 1.\end{aligned} \tag{38}$$

Since $\frac{\partial R_1(\lambda)}{\partial \lambda} < 0$ for all $\lambda \geq 0$, Eq. (38) does not have non-negative real roots if $R_1(0) < 1$, and has a real positive or trivial root if $R_1(0) \geq 1$. We arrive at Statement 2.

Statement 2. *If $R_1(0) < 1$, the disease-free equilibrium of system (1)–(13) is locally asymptotically stable, whereas if $R_1(0) \geq 1$ it is unstable.*

IV. Existence of endemic equilibrium

Endemic equilibrium of system (1)–(13):

$$S^*,\ V^*(a_1),\ L^*(a_2),\ I^*(a_3),\ A^*(a_4),\ R^*(a_5),$$

$$N_V^* = \int_0^{a_d^{(1)}} V^*(a_1)da_1,\ N_L^* = \int_0^{a_d^{(2)}} L^*(a_2)da_2,$$

$$N_I^* = \int_0^{a_d^{(3)}} I^*(a_3)da_3,\ N_A^* = \int_0^{a_d^{(4)}} A^*(a_4)da_4,$$

$$N_R^* = \int_0^{a_d^{(5)}} R^*(a_5)da_5,$$

satisfies the stationary system:

$$0 = -(\mu - b + q + f^*)S^* + V^*(a_d^{(1)}) + R^*(a_d^{(5)}), \tag{39}$$

$$\frac{dV^*(a_1)}{da_1} = -(\mu + (1-\sigma)f^*)V^*(a_1), \tag{40}$$

$$\frac{dL^*(a_2)}{da_2} = -\mu L^*(a_2), \tag{41}$$

$$\frac{dI^*(a_3)}{da_3} = -(\mu + \gamma)I^*(a_3), \tag{42}$$

$$\frac{dA^*(a_4)}{da_4} = -\mu A^*(a_4), \tag{43}$$

$$\frac{dR^*(a_5)}{da_5} = -(\mu + v)R^*(a_5), \tag{44}$$

$$f^* = \beta(N_I^* + \eta N_A^*), \tag{45}$$

with boundary conditions:

$$V^*(0) = qS^*, \tag{46}$$

$$L^*(0) = f^*(S^* + (1-\sigma)N_V^*), \tag{47}$$

$$I^*(0) = (1-\rho)L^*(a_d^{(2)}), \tag{48}$$

$$A^*(0) = \rho L^*(a_d^{(2)}), \tag{49}$$

$$R^*(0) = I^*(a_d^{(3)}) + A^*(a_d^{(4)}). \tag{50}$$

The formal solution of the stationary problem (39)–(50) is given by:

$$S^* = (\mu - b + q + f^*)^{-1}\big(V^*(a_d^{(1)}) + R^*(a_d^{(5)})\big), \tag{51}$$

$$V^*(a_1) = qS^* \exp(-(\mu + (1-\sigma)f^*)a_1), \tag{52}$$

$$L^*(a_2) = f^*(S^* + (1-\sigma)N_V^*) \exp(-\mu a_2), \tag{53}$$

$$I^*(a_3) = (1-\rho)L^*(a_d^{(2)}) \exp(-(\mu+\gamma)a_3), \tag{54}$$

$$A^*(a_4) = \rho L^*(a_d^{(2)}) \exp(-\mu a_4), \tag{55}$$

$$R^*(a_5) = (I^*(a_d^{(3)}) + A^*(a_d^{(4)})) \exp(-(\mu + v)a_5). \tag{56}$$

Substituting expressions for $V^*(a_d^{(1)})$, $I^*(a_d^{(3)})$, $A^*(a_d^{(4)})$, $R^*(a_d^{(5)})$, into the system (51)–(56) we arrive at the equations:

$$N_V^* = qS^*(\mu + (1-\sigma)f^*)^{-1} \times (1 - \exp(-(\mu + (1-\sigma)f^*)a_d^{(1)})), \tag{57}$$

$$N_L^* = L^*(a_d^{(2)}) \exp(\mu a_d^{(2)})\mu^{-1}(1 - \exp(-\mu a_d^{(2)})), \tag{58}$$

$$N_I^* = (1-\rho)L^*(a_d^{(2)})(\mu+\gamma)^{-1} \times (1 - \exp(-(\mu+\gamma)a_d^{(3)})) = C_I L^*(a_d^{(2)}), \tag{59}$$

$$C_I = (1-\rho)(\mu+\gamma)^{-1} \times (1 - \exp(-(\mu+\gamma)a_d^{(3)})) > 0, \tag{60}$$

$$N_A^* = \rho L^*(a_d^{(2)})\mu^{-1}(1 - \exp(-\mu a_d^{(4)})) = C_A L^*(a_d^{(2)}), \tag{61}$$

$$C_A = \rho\mu^{-1}(1 - \exp(-\mu a_d^{(4)})) > 0, \tag{62}$$

$$f^* = C_f L^*(a_d^{(2)}), \quad C_f = \beta(C_I + \eta C_A) > 0, \tag{63}$$

$$R^*(a_d^{(5)}) = C_R L^*(a_d^{(2)}), \tag{64}$$

$$C_R = ((1-\rho)\exp(-(\mu+\gamma)a_d^{(3)}) + \rho\exp(-\mu a_d^{(4)})) \times \exp(-(\mu+v)a_d^{(5)}), \quad 0 < C_R < 1, \tag{65}$$

$$N_R^* = C_R L^*(a_d^{(2)})(\mu+v)^{-1} \times (\exp((\mu+v)a_d^{(5)}) - 1). \tag{66}$$

Substituting Eqs. (63) in Eq. (53) taken at $a_d^{(2)}$, and substituting Eq. (57) in the obtained expression, after a little algebra, we arrive at the equation:

$$qS^* \exp(-(\mu + (1-\sigma)C_f L^*(a_d^{(2)}))a_d^{(1)}) = qS^* - (\mu + (1-\sigma)C_f L^*(a_d^{(2)})) \times (\exp(\mu a_d^{(2)}) - C_f S^*)((1-\sigma)C_f)^{-1}. \tag{67}$$

Substituting $V^*(a_d^{(1)})$ from (52), Eqs. (63) and (64) in Eq. (51) we have:

$$qS^* \exp(-(\mu + (1-\sigma)C_f L^*(a_d^{(2)}))a_d^{(1)}) \\ = (\mu - b + q + C_f L^*(a_d^{(2)}))S^* - C_R L^*(a_d^{(2)}). \quad (68)$$

Equating the right sides of Eqs. (67) and (68), after a little algebra we arrive to the linear equation for S^* and $L^*(a_d^{(2)})$:

$$S^* = C_L(L^*(a_d^{(2)}) + C_0), \quad (69)$$

$$C_L = (1-\sigma)(\exp(\mu a_d^{(2)}) - C_R) \\ \times (\sigma\mu + (1-\sigma)b)^{-1} > 0, \quad (70)$$

$$C_0 = \mu((1-\sigma)C_f(1 - \exp(-\mu a_d^{(2)})C_R))^{-1} > 0. \quad (71)$$

Substituting Eq. (69) in Eq. (68) yields the transcendental equation for $L^*(a_d^{(2)}) > 0$:

$$y_1(L^*(a_d^{(2)})) = y_2(L^*(a_d^{(2)})), \quad (72)$$

where:

$$y_1(L^*(a_d^{(2)})) = L^*(a_d^{(2)}) + C_0, \quad (73)$$

$$y_2(L^*(a_d^{(2)})) = p(L^*(a_d^{(2)}))g(L^*(a_d^{(2)})), \quad (74)$$

$$p(L^*(a_d^{(2)})) = C_f L^{*2}(a_d^{(2)}) + (\mu - b + q)C_0 \\ +(\mu - b + q + C_f C_0 - C_R C_L^{-1})L^*(a_d^{(2)}), \quad (75)$$

$$g(L^*(a_d^{(2)})) = q^{-1} \exp\big(a_d^{(1)} \\ \times (\mu + (1-\sigma)C_f L^*(a_d^{(2)}))\big). \quad (76)$$

We can also derive another valuable expression for the endemic equilibrium. Plugging Eqs. (57), (64), (65) into Eq. (39) yields:

$$S^* = \exp\big(-(\mu+\nu)a_d^{(5)}\big)L^*(a_d^{(2)})\times \\ \Big(\rho\exp\big(-\mu a_d^{(4)}\big) + (1-\rho)\exp\big(-(\mu+\gamma)a_d^{(3)}\big)\Big)\times \\ \Big(\mu - b + q(1 - \exp(-(\mu + (1-\sigma)f^*)a_d^{(1)})) + f^*\Big)^{-1} \quad (77)$$

On the other hand, plugging Eqs. (52), (54), (55), (56) into Eq. (39) yields:

$$S^*\Big(\mu - b + q(1 - \exp(-(\mu + (1-\sigma)f^*)a_d^{(1)})) + f^*\Big) \\ = (S^* + (1-\sigma)N_V^*)\exp\big(-\mu a_d^{(2)} - (\mu+\nu)a_d^{(5)}\big) \\ \times f^*\Big((1-\rho)\exp(-(\mu+\gamma)a_d^{(3)}) + \rho\exp(-(\mu)a_d^{(4)})\Big) \quad (78)$$

Plugging Eqs. (57), (63), (69), (70), (71), (77) into Eq. (78), after a little algebra we arrive to the final equation for endemic equilibrium S^* and f^* which is analogous of Eq. (38) for disease free equilibrium (we introduce here new parameter R_2):

$$R_2(0) = \beta S^*\bigg(1 + \frac{(1-\sigma)q}{(\mu + (1-\sigma)f^*)}\exp\big(-\mu a_d^{(2)}\big) \\ \times \Big(1 - \exp\big(-(\mu + (1-\sigma)f^*)a_d^{(1)}\big)\Big)\bigg) \\ \times \bigg((1-\rho)\frac{1 - \exp\big(-(\mu+\gamma)a_d^{(3)}\big)}{\mu+\gamma} \\ + \rho\eta\frac{1 - \exp\big(-\mu a_d^{(4)}\big)}{\mu}\bigg) = 1. \quad (79)$$

Thus, we arrive at Theorem 1.

Theorem 1. *If $R_0 > 1$, transcendental Eq. (72) has a unique real positive root $L^*(a_d^{(2)}) > 0$ and the system (1)–(13) possesses a unique endemic equilibrium. If $R_0 \le 1$, Eq. (72) does not have real positive roots and the system (1)–(13) does not have endemic equilibria. Endemic equilibrium is a stationary solution of system (1)–(13):*

$$S^*,\ V^*(a_1),\ L^*(a_2),\ I^*(a_3),\ A^*(a_4),\ R^*(a_5), \\ N_V^*,\ N_L^*,\ N_I^*,\ N_A^*,\ N_R^*,$$

which is defined uniquely through the root of transcendental Eq.(72) and Eqs. (51)–(66), (69)–(71).

Proof of Theorem 1 is given in Appendix B.

V. Local asymptotic stability of endemic equilibrium

Linearizing system (1)–(6) in the vicinity of endemic equilibrium:

$$S^* > 0,\ V^*(a_1) \ge 0,\ L^*(a_2) \ge 0, \\ I^*(a_3) \ge 0,\ A^*(a_4) \ge 0,\ R^*(a_5) \ge 0, \\ N_V^* > 0,\ N_L^* > 0,\ N_I^* > 0,\ N_A^* > 0,\ N_R^* > 0,$$

we arrive at the system for perturbations:

- for S^*:

$$\bar{\xi}_s(t) = \xi_s \exp(\lambda t)$$

- for $V^*(a_1)$ and N_V^*:

$$\bar{\psi}_v(a_1, t) = \psi_v(a_1)\exp(\lambda t)$$

$$\bar{\xi}_v(t) = \int_0^{a_d^{(1)}} \bar{\psi}_v(a_1, t)da_1 = \xi_v \exp(\lambda t)$$

- for $L^*(a_2)$ and N_L^*:

$$\bar{\psi}_l(a_2,t)=\psi_l(a_2)\exp(\lambda t)$$

$$\bar{\xi}_l(t)=\int_0^{a_d^{(2)}}\bar{\psi}_l(a_2,t)da_2=\xi_l\exp(\lambda t)$$

- for $I^*(a_3)$ and N_I^*:

$$\bar{\psi}_i(a_3,t)=\psi_i(a_3)\exp(\lambda t)$$

$$\bar{\xi}_i(t)=\int_0^{a_d^{(3)}}\bar{\psi}_i(a_3,t)da_3=\xi_i\exp(\lambda t)$$

- for $A^*(a_4)$ and N_A^*:

$$\bar{\psi}_a(a_4,t)=\psi_a(a_4)\exp(\lambda t)$$

$$\bar{\xi}_a(t)=\int_0^{a_d^{(4)}}\bar{\psi}_a(a_4,t)da_4=\xi_a\exp(\lambda t)$$

- for $R^*(a_5)$ and N_R^*:

$$\bar{\psi}_r(a_5,t)=\psi_r(a_5)\exp(\lambda t)$$

$$\bar{\xi}_r(t)=\int_0^{a_d^{(5)}}\psi_r(a_5,t)da_5=\xi_r\exp(\lambda t)$$

Then:

$$\begin{aligned}0=&-(\lambda+\mu-b+q+f^*)\xi_s-\beta(\xi_i+\eta\xi_a)S^*\\&+\psi_v(a_d^{(1)})+\psi_r(a_d^{(5)}),\end{aligned}\tag{80}$$

$$\begin{aligned}\frac{d\psi_v(a_1)}{da_1}=&-(\lambda+\mu+(1-\sigma)f^*)\psi_v(a_1)\\&-(1-\sigma)\beta(\xi_i+\eta\xi_a)V^*(a_1),\end{aligned}\tag{81}$$

$$\frac{d\psi_l(a_2)}{da_2}=-(\lambda+\mu)\psi_l(a_2),\tag{82}$$

$$\frac{d\psi_i(a_3)}{da_3}=-(\lambda+\mu+\gamma)\psi_i(a_3),\tag{83}$$

$$\frac{d\psi_a(a_4)}{da_4}=-(\lambda+\mu)\psi_a(a_4),\tag{84}$$

$$\frac{d\psi_r(a_5)}{da_5}=-(\lambda+\mu+\upsilon)\psi_r(a_5).\tag{85}$$

Eqs. (81)–(85) are completed by the following boundary conditions:

$$\psi_v(0)=q\xi_s,\tag{86}$$

$$\begin{aligned}\psi_l(0)=&\beta(S^*+(1-\sigma)N_V^*)(\xi_i+\eta\xi_a)\\&+f^*(\xi_s+(1-\sigma)\xi_v),\end{aligned}\tag{87}$$

$$\psi_i(0)=(1-\rho)\psi_l(a_d^{(2)}),\tag{88}$$

$$\psi_a(0)=\rho\psi_l(a_d^{(2)}),\tag{89}$$

$$\psi_r(0)=\psi_i(a_d^{(3)})+\psi_a(a_d^{(4)}).\tag{90}$$

Derivation of characteristic equation for stationary system (80)–(90) is given in Appendix C. Thus, characteristic equation for λ is:

$$Q(\lambda)=Z(\lambda),\tag{91}$$

$$Q(\lambda)=\big(w_1(\lambda)-w_2(\lambda)f^*\big)w_4(\lambda)w_5(\lambda),\tag{92}$$

$$Z(\lambda)=-w_3(\lambda)+w_2(\lambda)(S^*+(1-\sigma)N_V^*).\tag{93}$$

Here:

$$\begin{aligned}w_1(\lambda)=&\frac{\exp\big(\lambda(a_d^{(2)}+a_d^{(5)})\big)}{1+(1-\sigma)q\hat{I}_{v2}(a_d^{(1)},\lambda)}\Big(\lambda+\mu-b+f^*\\&+q\big(1-\exp\big(-(\lambda+\mu+(1-\sigma)f^*)a_d^{(1)}\big)\big)\Big),\end{aligned}\tag{94}$$

$$\begin{aligned}w_2(\lambda)=&\Big((1-\rho)\exp\big(-(\lambda+\mu+\gamma)a_d^{(3)}\big)\\&+\rho\exp\big(-(\lambda+\mu)a_d^{(4)}\big)\Big)\\&\times\exp\big(-\mu a_d^{(2)}-(\mu+\nu)a_d^{(5)}\big),\end{aligned}\tag{95}$$

$$\begin{aligned}w_3(\lambda)=&\frac{(1-\sigma)^2S^*\hat{I}_{v3}(a_d^{(1)},\lambda)\exp\big(\lambda(a_d^{(2)}+a_d^{(5)})\big)}{1+(1-\sigma)q\hat{I}_{v2}(a_d^{(1)},\lambda)}\\&\times\beta q\Big(\lambda+q\big(1-\exp\big(-(\lambda+\mu+(1-\sigma)f^*)a_d^{(1)}\big)\big)\\&+\mu-b+f^*\Big)+\beta S^*\Big(\exp\big(\lambda(a_d^{(2)}+a_d^{(5)})\big)\\&+(1-\sigma)q\exp\big(\lambda(a_d^{(2)}+a_d^{(5)}-a_d^{(1)})\big)\\&\times\exp\big(-(\mu+(1-\sigma)f^*)a_d^{(1)}\big)I_{v1}(a_d^{(1)},\lambda)\Big),\end{aligned}\tag{96}$$

$$w_4(\lambda)=1-R_2(\lambda),\tag{97}$$

$$\begin{aligned}R_2(\lambda)=&\beta S^*\exp\big(-(\lambda+\mu)a_d^{(2)}\big)\\&\times\Bigg(1+\frac{(1-\sigma)q}{\mu+(1-\sigma)f^*}\\&\times\Big(1-\exp\big(-(\mu+(1-\sigma)f^*)a_d^{(1)}\big)\Big)\Bigg)\\&\times\Bigg((1-\rho)\frac{1-\exp\big(-(\lambda+\mu+\gamma)a_d^{(3)}\big)}{\lambda+\mu+\gamma}\\&+\rho\eta\frac{1-\exp\big(-(\lambda+\mu)a_d^{(4)}\big)}{\lambda+\mu}\Bigg),\end{aligned}\tag{98}$$

$$\begin{aligned}w_5(\lambda)=&\big((1-\rho)I_i(a_d^{(3)},\lambda)+\eta\rho I_a(a_d^{(4)},\lambda)\big)^{-1}\\&\times\exp\big((\lambda+\mu)a_d^{(2)}\big),\end{aligned}\tag{99}$$

$$I_{v1}(a_1,\lambda) = \int_0^{a_1} \exp(\lambda\eta)d\eta = \frac{\exp(\lambda a_1)-1}{\lambda} > 0, \quad (100)$$

$$\hat{I}_{v2}(a_d^{(1)},\lambda) = \int_0^{a_d^{(1)}} \exp(-(\lambda+\mu+(1-\sigma)f^*)a_1)da_1$$
$$= \frac{1-\exp(-(\lambda+\mu+(1-\sigma)f^*)a_d^{(1)})}{\lambda+\mu+(1-\sigma)f^*} > 0, \quad (101)$$

$$\hat{I}_{v3}(a_d^{(1)},\lambda) = \int_0^{a_d^{(1)}} \exp(-(\lambda+\mu+(1-\sigma)f^*)a_1)$$
$$\times I_{v1}(a_1,\lambda)da_1 = \frac{1-\exp(-(\mu+(1-\sigma)f^*)a_d^{(1)})}{\lambda(\mu+(1-\sigma)f^*)}$$
$$-\frac{1-\exp(-(\lambda+\mu+(1-\sigma)f^*)a_d^{(1)})}{\lambda(\lambda+\mu+(1-\sigma)f^*)} > 0, \quad (102)$$

$$I_i(a_d^{(3)},\lambda) = \int_0^{a_d^{(3)}} \exp(-(\lambda+\mu+\gamma)a_3)da_3$$
$$= \frac{1-\exp(-(\lambda+\mu+\gamma)a_d^{(3)})}{\lambda+\mu+\gamma} > 0, \quad (103)$$

$$I_a(a_d^{(4)},\lambda) = \int_0^{a_d^{(4)}} \exp(-(\lambda+\mu)a_4)da_4$$
$$= \frac{1-\exp(-(\lambda+\mu)a_d^{(4)})}{\lambda+\mu} > 0. \quad (104)$$

The criterion of local asymptotic stability of endemic equilibrium is given by the following Theorem 2.

Theorem 2. *If $Z(0) < 0$, endemic equilibrium S^*, $V^*(a_1)$, $L^*(a_2)$, $I^*(a_3)$, $A^*(a_4)$, $R^*(a_5)$, N_V^*, N_L^*, N_I^*, N_A^*, N_R^* (Eqs. (33)–(48), (51)–(53)) is locally asymptotically stable, whereas if $Z(0) \geq 0$ it is unstable.*

Proof of Theorem 2 is given in Appendix D.

Remark 1. Using Eq. (93), we can obtain the criterion of asymptotic stability of endemic equilibrium $Z(0) < 0$ in unfolded form:

$$1 - C_R\exp(-\mu a_d^{(2)}) + (1-\sigma)\frac{q}{(\mu+(1-\sigma)f^*)}$$
$$\times\Big((1+(\mu+(1-\sigma)f^*)a_d^{(1)})$$
$$\times\exp(-(\mu+(1-\sigma)f^*)a_d^{(1)}) - C_R\exp(-\mu a_d^{(2)})\Big)$$
$$+(1-\sigma)^2 q\hat{I}_{v3}(a_d^{(1)},0)f^*C_R\exp(-\mu a_d^{(2)})$$
$$+(1-\sigma)^3 q^2\hat{I}_{v3}(a_d^{(1)},0)\hat{I}_{v2}(a_d^{(1)},0)$$
$$\times f^*C_R\exp(-\mu a_d^{(2)}) > 0, \quad (105)$$

where $C_R \in (0,1)$ is a dimensionless constant given by Eq. (65),

$$\hat{I}_{v2}(a_d^{(1)},0) = \frac{1-\exp\big(-(\mu+(1-\sigma)f^*)a_d^{(1)}\big)}{\mu+(1-\sigma)f^*} > 0,$$

$$\hat{I}_{v3}(a_d^{(1)},0) = \frac{\Big(1-(1+(\mu+(1-\sigma)f^*)a_d^{(1)})}{(\mu+(1-\sigma)f^*)^2}$$
$$\times\exp\big(-(\mu+(1-\sigma)f^*)a_d^{(1)}\big)\Big) > 0.$$

VI. Discussion and conclusions

This article is focused on the qualitative analysis of age-of-infection and -immunity structured SVLIAR-type model with 5 independent age variables for incubation period of COVID-19, period of vaccine immunity, period of COVID-19 disease treatment, period of asymptomatic COVID-19 disease and period of immunity after recovering. Either, there is only the disease-free equilibrium when the basic reproduction number equals to one $R_0 = 1$, or unique positive endemic equilibrium exists when the basic reproduction number is bigger than one $R_0 > 1$ [1,6,8,18].

The criterion of local asymptotic stability of disease-free equilibrium which controls the transition of system to the endemic or trivial equilibrium when disease-free equilibrium is unstable is derived. We proved that system has at most one endemic equilibrium when $R_0 > 1$.

Criterion of local asymptotic stability of such equilibrium is pretty complex, contains all coefficients of the model and relates the demographic characteristics of population (birth and death rates) with maximums of all age variables: $a_d^{(1)}$ (period of time when vaccination-induced immunity starts to wain (maximum age of vaccination-induced immunity)), $a_d^{(2)}$ (maximum incubation period of COVID-19 infection in organism), $a_d^{(3)}$ (maximum period of infectious disease (or disease treatment)), $a_d^{(4)}$ (maximum period of asymptomatic infectious disease), $a_d^{(5)}$ is a period when disease-induced immunity of individuals starts to wain after recovering (maximum age of disease-induced immunity), disease-induced death rate, death rate induced by the complications after COVID-19 disease, fraction of fully vaccinated susceptibles per unit of time, vaccination efficacy, rate a coefficient of modification of disease transmission and fraction of asymptomatic individuals without disease symptoms.

This result is a direct extension of similar results for unstructured epidemic SVLIAR model [14,15]. Our results show that efficacy of vaccination and fraction of vaccinated susceptibles play a crucial role in stabilization of COVID-19 disease transmission among population. Thus, similarly to unstructured epidemic SVLIAR models, vaccination can cause asymptotic stability of endemic equilibrium.

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Appendix A. Characteristic equation of disease-free equilibrium

System (27)–(37) is reduced to the homogeneous system of linear algebraic equations for perturbations:

$$(\lambda+\mu-b+q)\xi_{s0} = -\beta S_0(\xi_{i0}+\eta\xi_{a0}) + \psi_{v0}(a_d^{(1)}) + \psi_{r0}(a_d^{(5)}) \tag{106}$$

$$\psi_{v0}(a_1) = q\exp(-(\lambda+\mu)a_1)\xi_{s0} - (1-\sigma)\beta q S_0 \exp(-(\lambda+\mu)a_1) I_{v1}(a_1,\lambda)(\xi_{i0}+\eta\xi_{a0}) \tag{107}$$

$$I_{v1}(a_1,\lambda) = \int_0^{a_1} \exp(\lambda\eta)d\eta = \frac{\exp(\lambda a_1)-1}{\lambda} > 0$$

$$\xi_{v0} = qI_{v2}(a_d^{(1)},\lambda)\xi_{s0} - (1-\sigma)\beta q S_0 I_{v3}(a_d^{(1)},\lambda)(\xi_{i0}+\eta\xi_{a0}) \tag{108}$$

$$I_{v2}(a_d^{(1)},\lambda) = \int_0^{a_d^{(1)}} \exp(-(\lambda+\mu)a_1)da_1 = \frac{1-\exp(-(\lambda+\mu)a_d^{(1)})}{\lambda+\mu} > 0$$

$$I_{v3}(a_d^{(1)},\lambda) = \int_0^{a_d^{(1)}} \exp(-(\lambda+\mu)a_1) I_{v1}(a_1,\lambda)da_1 = \frac{1}{\lambda}\Big(\frac{1-\exp(-\mu a_d^{(1)})}{\mu} - \frac{1-\exp(-(\lambda+\mu)a_d^{(1)})}{\lambda+\mu}\Big) > 0$$

$$\psi_{l0}(a_2) = \exp(-(\lambda+\mu)a_2)\beta(S_0+(1-\sigma)N_{V0}^*)(\xi_{i0}+\eta\xi_{a0}) \tag{109}$$

$$\psi_{i0}(a_3) = \exp(-(\lambda+\mu+\gamma)a_3)\exp(-(\lambda+\mu)a_d^{(2)})(1-\rho)\beta(S_0+(1-\sigma)N_{V0}^*)(\xi_{i0}+\eta\xi_{a0}) \tag{110}$$

$$\xi_{i0} = I_i(a_d^{(3)},\lambda)\exp(-(\lambda+\mu)a_d^{(2)})(1-\rho)\beta(S_0+(1-\sigma)N_{V0}^*)(\xi_{i0}+\eta\xi_{a0}) \tag{111}$$

$$I_i(a_d^{(3)},\lambda) = \int_0^{a_d^{(3)}} \exp(-(\lambda+\mu+\gamma)a_3)da_3 = \frac{1-\exp(-(\lambda+\mu+\gamma)a_d^{(3)})}{\lambda+\mu+\gamma} > 0$$

$$\psi_{a0}(a_4) = \exp(-(\lambda+\mu)a_4)\exp(-(\lambda+\mu)a_d^{(2)})\rho\beta(S_0+(1-\sigma)N_{V0}^*)(\xi_{i0}+\eta\xi_{a0}) \tag{112}$$

$$\xi_{a0} = I_a(a_d^{(4)},\lambda)\exp(-(\lambda+\mu)a_d^{(2)})\rho\beta(S_0+(1-\sigma)N_{V0}^*)(\xi_{i0}+\eta\xi_{a0}) \tag{113}$$

$$I_a(a_d^{(4)},\lambda) = \int_0^{a_d^{(4)}} \exp(-(\lambda+\mu)a_4)da_4 = \frac{1-\exp(-(\lambda+\mu)a_d^{(4)})}{\lambda+\mu} > 0$$

$$\begin{aligned}\psi_{r0}(a_5) &= \exp(-(\lambda+\mu+\nu)a_5)((1-\rho)\exp(-(\lambda+\mu+\gamma)a_d^{(3)}) + \rho\exp(-(\lambda+\mu)a_d^{(4)})) \\ &\quad\times \exp(-(\lambda+\mu)a_d^{(2)})(S_0+(1-\sigma)N_{V0}^*)(\xi_{i0}+\eta\xi_{a0})\end{aligned} \tag{114}$$

Plugging Eqs. (25), (107), (114) into Eq. (106) we arrive to the equation for ψ_{s0}:

$$\begin{aligned}\lambda\xi_{s0} &= S_0(-\beta(1+(1-\sigma)q\exp(-(\lambda+\mu)a_d^{(1)})I_{v1}(a_d^{(1)},\lambda)) + \eta\exp(-(\lambda+\mu+\nu)a_d^{(5)}) \\ &\quad\times \exp(-(\lambda+\mu)a_d^{(2)})((1-\rho)\exp(-(\lambda+\mu+\gamma)a_d^{(3)}) + \rho\exp(-(\lambda+\mu)a_d^{(4)})) \\ &\quad\times (1+(1-\sigma)\frac{q}{\mu}(1-\exp(-\mu a_d^{(1)}))))(\xi_{i0}+\eta\xi_{a0}).\end{aligned} \tag{115}$$

Since Eqs. (107), (109), (110), (112), (114), (115) depend linearly, without singularities from linear combination of perturbations $(\xi_{i0}+\eta\xi_{a0})$, it is sufficient to analyze here only equations (111) and (113). Multiplying Eq. (113) by constant $\eta>0$ and summarizing it with Eq. (111), after a little algebra we arrive to the characteristic equation of DFE:

$$\begin{aligned}R_1(\lambda) &= \beta S_0\Big(1+(1-\sigma)\frac{q}{\mu}(1-\exp(-\mu a_d^{(1)}))\Big)\exp(-(\lambda+\mu)a_d^{(2)}) \\ &\quad\times\Big((1-\rho)\frac{1-\exp(-(\lambda+\mu+\gamma)a_d^{(3)})}{\lambda+\mu+\gamma} + \rho\eta\frac{1-\exp(-(\lambda+\mu)a_d^{(4)})}{\lambda+\mu}\Big) = 1.\end{aligned} \tag{116}$$

Appendix B. Proof of Theorem 1

1. If $R_0 > 1$, we have $(\mu - b + q)q^{-1}\exp(\mu a_d^{(1)}) < 1$ and $y_1(0) > y_2(0)$ (Eqs. (74), (75)), where:

$$y_1(0) = C_0, \tag{117}$$

$$y_2(0) = (\mu - b + q)q^{-1}\exp(\mu a_d^{(1)})C_0. \tag{118}$$

From the properties of elementary algebraic functions, we get the following properties of $y_1(x)$, $y_2(x)$:

$$\lim_{x\to-\infty} y_1 = -\infty,\ \lim_{x\to\infty} y_1 = \infty, \tag{119}$$

$$\lim_{x\to-\infty} y_2 = 0,\ \lim_{x\to\infty} y_2 = \infty, \tag{120}$$

$$\lim_{x\to\infty} \frac{y_1(x)}{y_2(x)} = 0. \tag{121}$$

On the other hand,

$$y_1^{'}(L^*(a_d^{(2)})) = 1, \tag{122}$$

$$\begin{aligned} y_2^{'}(L^*(a_d^{(2)})) &= p'(L^*(a_d^{(2)}))g(L^*(a_d^{(2)})) + p(L^*(a_d^{(2)}))g'(L^*(a_d^{(2)})) \\ &= (F_1 L^{*2}(a_d^{(2)}) + (F_2+F_3)L^*(a_d^{(2)}) + F_4)q^{-1}(1-\sigma)a_d^{(1)}C_f\exp((\mu + (1-\sigma)C_f L^*(a_d^{(2)}))a_d^{(1)}), \end{aligned} \tag{123}$$

where:

$$F_1 = C_f, \tag{124}$$

$$F_2 = \mu - b + q + C_f C_0 - C_R C_L^{-1}, \tag{125}$$

$$F_3 = 2((1-\sigma)a_d^{(1)})^{-1}, \tag{126}$$

$$F_4 = F_2((1-\sigma)a_d^{(1)}F_1)^{-1}. \tag{127}$$

Equation $y_2^{'}(L^*(a_d^{(2)})) = 0$ has two roots $\bar{L}_{1,2}^*(a_d^{(2)})$:

$$\bar{L}_1^*(a_d^{(2)}) = (2F_1)^{-1}\Big(-F_2 - F_3 - \sqrt{F_2^2 + F_3^2}\Big), \tag{128}$$

$$\bar{L}_2^*(a_d^{(2)}) = (2F_1)^{-1}\Big(-F_2 - F_3 + \sqrt{F_2^2 + F_3^2}\Big). \tag{129}$$

Since $F_3 > 0$, the first root is always negative $\bar{L}_1^*(a_d^{(2)}) < 0$. The second root is negative $\bar{L}_2^*(a_d^{(2)}) < 0$ if $F_2 > 0$, and it is non-negative $\bar{L}_2^*(a_d^{(2)}) \geq 0$ if $F_2 \leq 0$. From Eq. (120) it follows that function $y_2(L^*(a_d^{(2)}))$ has a relative maximum at $\bar{L}_1^*(a_d^{(2)}) < 0$ and a relative minimum at $\bar{L}_2^*(a_d^{(2)})$. If $\bar{L}_2^*(a_d^{(2)}) < 0$ derivative $y_2^{'}(L^*(a_d^{(2)})) > 0$ for all $L^*(a_d^{(2)}) \geq 0$. If $\bar{L}_2^*(a_d^{(2)}) \geq 0$ derivative $y_2^{'}(0) \leq 0$, $y_2^{'}(L^*(a_d^{(2)})) \geq 0$ for all $L^*(a_d^{(2)}) \geq \bar{L}_2^*(a_d^{(2)})$. Thus, from Eq. (121) it follows that if $y_1(0) > y_2(0)$, in all cases ($\bar{L}_2^*(a_d^{(2)}) < 0$ or $\bar{L}_2^*(a_d^{(2)}) \geq 0$) functions $y_1(L^*(a_d^{(2)}))$ and $y_2(L^*(a_d^{(2)}))$ always intersect only one time at some positive point $L^*(a_d^{(2)}) > 0$, characteristic Eq.(73) has a unique real positive root $L^*(a_d^{(2)}) > 0$, and system (1)–(13) possesses a unique endemic equilibrium S^*, $V^*(a_1)$, $L^*(a_2)$, $I^*(a_3)$, $A^*(a_4)$, $R^*(a_5)$, N_V^*, N_L^*, N_I^*, N_A^*, N_R^* defined uniquely through the root of transcendental Eq.(73) and Eqs. (52)–(67), (70)–(72).

2. If $R_0 \leq 1$, then $(\mu - b + q)q^{-1}\exp(\mu a_d^{(1)}) \geq 1$ and $y_1(0) \leq y_2(0)$. We can show that in this case $y_2^{'}(0) \geq y_1^{'}(0) = 1$:

$$y_2^{'}(0) = (\mu - b + q + C_f C_0 - C_R C_L^{-1} + (\mu - b + q)C_f C_0(1-\sigma)a_d^{(1)})q^{-1}\exp(\mu a_d^{(1)}) > 1, \tag{130}$$

or

$$\mu - b + q(1 - \exp(-\mu a_d^{(1)})) + C_f C_0 - C_R C_L^{-1} + (\mu - b + q)C_f C_0(1-\sigma)a_d^{(1)} > 0. \tag{131}$$

Using Eqs. (122), (123) and (61), (63), (64), (66), (71), (72), after a little algebra we can transform inequality (130) (or (131)) to:

$$g_1(\mu) < g_2(\mu) + g_3(\mu), \tag{132}$$

where

$$g_1(\mu) = (1 + (1-\sigma)\mu^{-1} q(1-\exp(-\mu a_d^{(1)})))((1-\rho)\exp(-(\mu+\gamma)a_d^{(3)}) + \rho\exp(-\mu a_d^{(1)})), \tag{133}$$

$$g_2(\mu) = (1 + (1-\sigma)a_d^{(1)} q\exp(-\mu a_d^{(1)}))\exp(\mu a_d^{(2)} + (\mu+\gamma)a_d^{(5)}), \tag{134}$$

$$g_3(\mu) = (\mu - b + q(1-\exp(-\mu a_d^{(1)})))(1-\sigma)(a_d^{(1)} + \mu^{-1})\exp(\mu a_d^{(2)} + (\mu+\gamma)a_d^{(5)}). \tag{135}$$

If $R_0 = 1$, then $\mu - b + q(1-\exp(-\mu a_d^{(1)})) = 0$, function $g_3(\mu) = 0$. We have:

$$\lim_{\mu\to 0} g_1(\mu) = (1 + (1-\sigma)q a_d^{(1)})((1-\rho)\exp(-\gamma a_d^{(3)}) + \rho), \tag{136}$$

$$\lim_{\mu\to 0} g_2(\mu) = (1 + (1-\sigma)q a_d^{(1)})\exp(\gamma a_d^{(5)}). \tag{137}$$

Since $0 < \rho < 1$, we have $\lim_{\mu\to 0} g_1(\mu) < \lim_{\mu\to 0} g_2(\mu)$. On the other hand, from $\mu a_d^{(2)} + (\mu+\gamma)a_d^{(5)} \geq \mu a_d^{(1)}$, $g_1^{'}(\mu) < 0$, $g_2^{'}(\mu) > 0$ for all $\mu > 0$ follows inequality (132), (130) and $y_2^{'}(0) \geq y_1^{'}(0) = 1$.

If $R_0 < 1$, then $\mu - b + q(1-\exp(-\mu a_d^{(1)})) > 0$, $g_3(\mu) > 0$, and, we arrive to inequality (132), (130) and $y_2^{'}(0) \geq y_1^{'}(0) = 1$.

Thus, since $y_2^{'}(0) \geq 1$ root $\bar{L}_2^*(a_d^{(2)}) < 0$, and $F_2 = \mu - b + q + C_f C_0 - C_R C_L^{-1} > 0$. From Eq. (123) it follows, that in this case $y_2^{'}(L^*(a_d^{(2)})) > y_2^{'}(0) \geq y_1^{'}(0) = 1$ for all $L^*(a_d^{(2)}) > 0$. Since $y_1(0) \leq y_2(0)$, functions $y_1(L^*(a_d^{(2)}))$ and $y_2(L^*(a_d^{(2)}))$ never intersect at positive point $L^*(a_d^{(2)}) > 0$, characteristic Eq. (73) does not have real positive roots, and system (1)–(13) does not have endemic equilibrium. Theorem 1 is proved.

Appendix C. Characteristic equation of endemic equilibrium

System (80)–(90) is reduced to the homogeneous system of linear algebraic equations for perturbations:

$$(\lambda + \mu - b + q + f^*)\xi_s = -\beta S^*(\xi_i + \eta\xi_a) + \psi_v(a_d^{(1)}) + \psi_r(a_d^{(5)}) \tag{138}$$

$$\begin{aligned}\psi_v(a_1) &= q\exp(-(\lambda+\mu+(1-\sigma)f^*)a_1)\xi_s \\ &\quad - (1-\sigma)\beta q S^*\exp(-(\lambda+\mu+(1-\sigma)f^*)a_1)I_{v1}(a_1,\lambda)(\xi_i+\eta\xi_a)\end{aligned} \tag{139}$$

$$\xi_v = q\hat{I}_{v2}(a_d^{(1)},\lambda)\xi_s - (1-\sigma)\beta q S^*\hat{I}_{v3}(a_d^{(1)},\lambda)(\xi_i+\eta\xi_a) \tag{140}$$

$I_{v1}(a_1,\lambda)$ is the same as in Appendix A

$$\hat{I}_{v2}(a_d^{(1)},\lambda) = \int_0^{a_d^{(1)}} \exp(-(\lambda+\mu+(1-\sigma)f^*)a_1)da_1 = \frac{1-\exp(-(\lambda+\mu+(1-\sigma)f^*)a_d^{(1)})}{\lambda+\mu+(1-\sigma)f^*} > 0$$

$$\begin{aligned}\hat{I}_{v3}(a_d^{(1)},\lambda) &= \int_0^{a_d^{(1)}} \exp(-(\lambda+\mu+(1-\sigma)f^*)a_1)I_{v1}(a_1,\lambda)da_1 \\ &= \frac{1-\exp(-(\mu+(1-\sigma)f^*)a_d^{(1)})}{\lambda(\mu+(1-\sigma)f^*)} - \frac{1-\exp(-(\lambda+\mu+(1-\sigma)f^*)a_d^{(1)})}{\lambda(\lambda+\mu+(1-\sigma)f^*)} > 0\end{aligned}$$

$$\psi_l(a_2) = \exp(-(\lambda+\mu)a_2)(\beta(S^* + (1-\sigma)N_V^*)(\xi_i+\eta\xi_a) + f^*(\xi_s + (1-\sigma)\xi_v)) \tag{141}$$

$$\begin{aligned}\psi_i(a_3) &= \exp(-(\lambda+\mu+\gamma)a_3)\exp(-(\lambda+\mu)a_d^{(2)})(1-\rho)(\beta(S^* + (1-\sigma)N_V^*) \\ &\quad \times (\xi_i+\eta\xi_a) + f^*(\xi_s + (1-\sigma)\xi_v))\end{aligned} \tag{142}$$

$$\xi_i = I_i(a_d^{(3)},\lambda)\exp(-(\lambda+\mu)a_d^{(2)})(1-\rho)(\beta(S^* + (1-\sigma)N_V^*)(\xi_i+\eta\xi_a) + f^*(\xi_s + (1-\sigma)\xi_v)) \tag{143}$$

$$\psi_a(a_4) = \exp(-(\lambda+\mu)a_4)\exp(-(\lambda+\mu)a_d^{(2)})\rho(\beta(S^* + (1-\sigma)N_V^*)(\xi_i+\eta\xi_a) + f^*(\xi_s + (1-\sigma)\xi_v)) \tag{144}$$

$$\xi_a = I_a(a_d^{(4)},\lambda)\exp(-(\lambda+\mu)a_d^{(2)})\rho(\beta(S^*+(1-\sigma)N_V^*)(\xi_i+\eta\xi_a)+f^*(\xi_s+(1-\sigma)\xi_v)) \tag{145}$$

$I_i(a_d^{(3)},\lambda)$ is the same as in Appendix A

$I_a(a_d^{(4)},\lambda)$ is the same as in Appendix A

$$\begin{aligned}\psi_r(a_5) &= \exp(-(\lambda+\mu+\nu)a_5)((1-\rho)\exp(-(\lambda+\mu+\gamma)a_d^{(3)})+\rho\exp(-(\lambda+\mu)a_d^{(4)}))\\ &\times\exp(-(\lambda+\mu)a_d^{(2)})(\beta(S^*+(1-\sigma)N_V^*)(\xi_i+\eta\xi_a)+f^*(\xi_s+(1-\sigma)\xi_v))\end{aligned} \tag{146}$$

Plugging Eqs. (139), (146) into Eq. (138), after a little algebra we have:

$$\begin{aligned}&\Big(\lambda+\mu-b+q(1-\exp(-(\lambda+\mu+(1-\sigma)f^*)a_d^{(1)}))+f^*\Big)(\xi_s+(1-\sigma)\xi_v)\\ &-\Big(\lambda+\mu-b+q(1-\exp(-(\lambda+\mu+(1-\sigma)f^*)a_d^{(1)}))+f^*\Big)(1-\sigma)\xi_v\\ &=\Big(-\beta S^*(1+(1-\sigma)q\exp(-(\lambda+\mu+(1-\sigma)f^*)a_d^{(1)})I_{v1}(a_d^{(1)},\lambda))+\exp(-(\lambda+\mu+\nu)a_d^{(5)})\\ &\times\exp(-(\lambda+\mu)a_d^{(2)})((1-\rho)\exp(-(\lambda+\mu+\gamma)a_d^{(3)})+\rho\exp(-(\lambda+\mu)a_d^{(4)}))\beta(S^*+(1-\sigma)N_V^*)\Big)(\xi_i+\eta\xi_a)\\ &+\exp(-(\lambda+\mu+\nu)a_d^{(5)})\exp(-(\lambda+\mu)a_d^{(2)})f^*(\xi_s+(1-\sigma)\xi_v)\\ &\times\Big((1-\rho)\exp(-(\lambda+\mu+\gamma)a_d^{(3)})+\rho\exp(-(\lambda+\mu)a_d^{(4)})\Big).\end{aligned} \tag{147}$$

Eq. (140) can be transformed to the linear equation:

$$\xi_v = \frac{q\hat{I}_{v2}(a_d^{(1)},\lambda)}{1+(1-\sigma)q\hat{I}_{v2}(a_d^{(1)},\lambda)}(\xi_s+(1-\sigma)\xi_v)-\frac{(1-\sigma)\beta qS^*\hat{I}_{v3}(a_d^{(1)},\lambda)}{1+(1-\sigma)q\hat{I}_{v2}(a_d^{(1)},\lambda)}(\xi_i+\eta\xi_a). \tag{148}$$

Equations (143), (145) give the linear equation:

$$\Big(((1-\rho)I_i(a_d^{(3)},\lambda)+\eta\rho I_a(a_d^{(4)},\lambda))\exp(-(\lambda+\mu)a_d^{(2)})f^*\Big)(\xi_s+(1-\sigma)\xi_v)=(1-R_1^S(\lambda))(\xi_i+\eta\xi_a). \tag{149}$$

After linear transformations of Eqs. (147), (148) and using Eq. (149), we arrive to the linear system for perturbations $(\xi_s+(1-\sigma)\xi_v)$ and $(\xi_i+\eta\xi_a)$. Since all perturbations can be defined through $(\xi_s+(1-\sigma)\xi_v)$ and $(\xi_i+\eta\xi_a)$, the characteristic equation is deriving for such linear system:

$$(w_1(\lambda)-w_2(\lambda)f^*)(\xi_s+(1-\sigma)\xi_v)+(w_3(\lambda)-w_2(\lambda)\beta(S^*+(1-\sigma)N_V^*))(\xi_i+\eta\xi_a)=0, \tag{150}$$

$$w_4(\lambda)f^*(\xi_s+(1-\sigma)\xi_v)-w_5(\lambda)(\xi_i+\eta\xi_a)=0, \tag{151}$$

where:

$$w_1(\lambda)=\frac{\Big(\lambda+\mu-b+q\big(1-\exp(-(\lambda+\mu+(1-\sigma)f^*)a_d^{(1)})\big)+f^*\Big)\exp\big(\lambda(a_d^{(2)}+a_d^{(5)})\big)}{1+(1-\sigma)q\hat{I}_{v2}(a_d^{(1)},\lambda)}, \tag{152}$$

$$w_2(\lambda)=\exp\big(-\mu a_d^{(2)}-(\mu+\nu)a_d^{(5)}\big)\Big((1-\rho)\exp(-(\lambda+\mu+\gamma)a_d^{(3)})+\rho\exp(-(\lambda+\mu)a_d^{(4)})\Big), \tag{153}$$

$$\begin{aligned}w_3(\lambda)&=\frac{\Big(\lambda+\mu-b+q\big(1-\exp(-(\lambda+\mu+(1-\sigma)f^*)a_d^{(1)})\big)+f^*\Big)\exp\big(\lambda(a_d^{(2)}+a_d^{(5)})\big)}{1+(1-\sigma)q\hat{I}_{v2}(a_d^{(1)},\lambda)}\\ &\times(1-\sigma)^2\beta qS^*\hat{I}_{v3}(a_d^{(1)},\lambda)+\beta S^*\Big(\exp(\lambda(a_d^{(2)}+a_d^{(5)}))\\ &+(1-\sigma)q\exp(\lambda(a_d^{(2)}+a_d^{(5)}-a_d^{(1)}))\exp(-(\mu+(1-\sigma)f^*)a_d^{(1)})I_{v1}(a_d^{(1)},\lambda)\Big),\end{aligned} \tag{154}$$

$$w_4(\lambda)=\big((1-\rho)I_i(a_d^{(3)},\lambda)+\eta\rho I_a(a_d^{(4)},\lambda)\big)\exp(-(\lambda+\mu)a_d^{(2)}), \tag{155}$$

$$w_5(\lambda)=1-R_1^S(\lambda). \tag{156}$$

Equating to zero determinant of this system we obtain finally the characteristic equation of system (80)–(90):

$$y_3(\lambda)=y_4(\lambda), \tag{157}$$

where:

$$y_3(\lambda) = (w_1(\lambda) - w_2(\lambda)f^*)w_5(\lambda), \tag{158}$$

$$y_4(\lambda) = -(w_3(\lambda) - w_2(\lambda)(S^* + (1-\sigma)N_V^*))w_4(\lambda)f^*. \tag{159}$$

APPENDIX D. PROOF OF THEOREM 2

It is easy to verify that supplementary functions satisfy inequalities:

$$\frac{\partial I_{v1}(a_1,\lambda)}{\partial\lambda} = \int\limits_0^{a_1} \eta\exp(\lambda\eta)d\eta > 0, \tag{160}$$

$$\frac{\partial \hat{I}_{v2}(a_d^{(1)},\lambda)}{\partial\lambda} = -\int\limits_0^{a_d^{(1)}} a_1\exp(-(\lambda+\mu+(1-\sigma)f^*)a_1)da_1 < 0, \tag{161}$$

$$\frac{\partial \hat{I}_{v3}(a_d^{(1)},\lambda)}{\partial\lambda} = \int\limits_0^{a_d^{(1)}} \exp(-(\mu+(1-\sigma)f^*)a_1)\frac{\partial I_{v1}(a_1,\lambda)}{\partial\lambda}da_1 > 0, \tag{162}$$

$$\frac{\partial I_i(a_d^{(3)},\lambda)}{\partial\lambda} = -\int\limits_0^{a_d^{(3)}} a_3\exp(-(\lambda+\mu+\gamma)a_3)da_3 < 0, \tag{163}$$

$$\frac{\partial I_a(a_d^{(4)},\lambda)}{\partial\lambda} = -\int\limits_0^{a_d^{(4)}} a_4\exp(-(\lambda+\mu)a_4)da_4 < 0. \tag{164}$$

From the properties of elementary functions it follows that for all $\lambda \geq 0$: function $w_1(\lambda) > 0$ and $\frac{dw_1(\lambda)}{d\lambda} > 0$, function $w_2(\lambda) > 0$ and $\frac{dw_2(\lambda)}{d\lambda} < 0$, function $w_5(\lambda) > 0$ and $\frac{dw_5(\lambda)}{d\lambda} > 0$. From Eq. (78) it follows that:

$$w_1(0) - w_2(0)f^* = 0,\ \frac{d(w_1(\lambda) - w_2(\lambda)f^*)}{d\lambda} > 0 \ \text{ and } \ w_1(\lambda) - w_2(\lambda)f^* > 0 \ \text{ for all } \ \lambda > 0.$$

Since $R_2(0) = 1$ (Eq. (80)), $\frac{\partial R_2(\lambda)}{\partial\lambda} < 0$ and $\lim_{\lambda\to\infty} R_2(\lambda) = 0$, function $w_4(0) = 0$, $w_4(\lambda) > 0$ and $\frac{dw_4(\lambda)}{d\lambda} > 0$ for all $\lambda \geq 0$. Thus, $Q(0) = 0$, $\frac{dQ(\lambda)}{d\lambda} > 0$ for all $\lambda \geq 0$, and $\lim_{\lambda\to\infty} Q(\lambda) = \infty$, i.e. $Q(\lambda)$ is an increasing from zero to infinity monotonic non-negative function for all $\lambda \geq 0$.

On the other hand, for all $\lambda \geq 0$: function $w_3(\lambda) > 0$ and $\frac{dw_3(\lambda)}{d\lambda} > 0$. Taking into account properties of $w_2(\lambda)$, equation (78) and inequality $a_d^{(2)} + a_d^{(5)} > a_d^{(1)}$ we obtain the properties of $Z(\lambda)$: $\frac{dZ(\lambda)}{d\lambda} < 0$ for all $\lambda \geq 0$ and$\lim_{\lambda\to\infty} Z(\lambda) = -\infty$, i.e. $Z(\lambda)$ is decreasing to negative infinity monotonic function.

Obviously, that $Q(\lambda)$ and $Z(\lambda)$ intersect at some point $\lambda^* \geq 0$ if and only if $Z(0) \geq Q(0) = 0$. In this case λ^* is a real non-negative root of characteristic equation (91) and endemic equilibrium is unstable. And, vice versa, if $Z(0) < Q(0) = 0$, functions $Q(\lambda)$ and $Z(\lambda)$ do not intersect at some point $\lambda^* \geq 0$, characteristic equation (91) does not have non-negative real root and endemic equilibrium is locally asymptotically stable. Theorem 2 is proved.