

Bifurcation-aware optimal control of discrete SIRS epidemic models: suppressing period-doubling and multi-wave outbreaks

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Abstract: Discrete-time epidemic models may display rich dynamics, including oscillatory outbreaks and chaotic behavior arising through period-doubling (Flip) bifurcations when key parameters, such as transmission or recruitment rates, vary. In most optimal control studies, the main objective is to reduce infection prevalence and intervention cost, while the possible impact of bifurcation-induced qualitative changes is often left aside. In this paper, we propose a *bifurcation-aware* optimal control framework for a discrete-time SIRS epidemic model that aims not only to reduce infection levels and control effort, but also to suppress multi-wave epidemic behavior. We first study the equilibria of the controlled system and derive the corresponding controlled basic reproduction number. We then analyze the local stability of the endemic equilibrium through the Jacobian spectrum and introduce a practical Flip-proximity indicator, based on the distance of eigenvalues to -1 , in order to identify regimes that are close to period-doubling. Motivated by this analysis, we formulate a discrete-time optimal control problem with two intervention mechanisms: modulation of population recruitment and reduction of disease transmission. To discourage persistent oscillatory behavior in the infected population, we include an additional oscillation-penalty term in the objective functional, which serves as a control-oriented surrogate for limiting higher-period outbreak patterns. The resulting optimality system is derived by means of a discrete Pontryagin maximum principle and solved numerically using a projected forward-backward algorithm under bounded control constraints. Numer-

ical results indicate that the proposed strategy leads to smoother infection profiles, lower peak prevalence, and a greater tendency to avoid flip-prone regimes than standard quadratic-cost control formulations.

Keywords: discrete SIRS model, flip bifurcation, period-doubling, epidemic oscillations, bifurcation-aware optimal control, discrete Pontryagin principle, chaos suppression

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I. INTRODUCTION

Mathematical modeling has long been an essential tool for understanding how infectious diseases spread and for designing effective intervention strategies. Among the most widely used approaches in mathematical epidemiology are compartmental models such as the classical SIR and SIRS systems, which describe the movement of individuals through susceptible, infected, and recovered classes while accounting for mechanisms such as recovery, loss of immunity, and demographic turnover [1]. These models have been studied extensively in both continuous and discrete time. In particular, discrete-time formulations are especially relevant when epidemiological data are recorded and reported at fixed intervals, as is often the case in practice.

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Unlike their continuous-time counterparts, discrete-time epidemic models may display a broader range of dynamical behaviors, including stable oscillations, multi-wave outbreaks, and even chaotic dynamics generated through bifurcation phenomena. Early work on discrete SIR-type systems already highlighted the importance of threshold conditions and local stability [2–4]. More recent studies have shown that discrete SIR and SIRS models can generate recurrent oscillatory patterns even when parameters remain constant [5, 6]. These observations suggest that classical epidemiological analysis based only on equilibria and threshold numbers is not always sufficient, and that tools from discrete dynamical systems are also needed.

In this context, bifurcation theory provides a natural framework for describing qualitative changes in system behavior as parameters vary. Of particular relevance in discrete-time models are fold and flip bifurcations, which can mark the onset of sudden transitions and oscillatory behavior. Foundational contributions by Carcassès [7, 8] developed systematic methods for analyzing fold and flip bifurcation curves in discrete maps, and related ideas have since been applied to epidemic models, where period-doubling has been shown to trigger cascades leading to recurrent and sometimes chaotic outbreak patterns [3, 4, 6]. From an epidemiological viewpoint, this is important because repeated epidemic waves may arise not only from seasonal forcing or exogenous shocks, but also from the intrinsic dynamics of the discrete model itself.

The COVID-19 pandemic further highlighted the importance of discrete and data-driven epidemic modeling. Real epidemic time series often exhibit substantial variability due to behavioral changes, reporting noise, environmental effects, and shifting intervention policies [9, 10]. Publicly available datasets, including those from the World Bank and related COVID-19 repositories [11], also reveal repeated epidemic waves that are not fully explained by equilibrium-based reasoning alone. These features motivate the search for control strategies that address not only disease persistence, but also oscillatory and multi-wave epidemic behavior.

Optimal control theory has been widely used to design intervention policies that reduce disease burden while balancing implementation costs. By using Pontryagin-type methods, many studies have proposed vaccination, treatment, distancing, and related intervention strategies for epidemic systems, including COVID-19 models [12–15]. However, most of these formulations are essentially prevalence-driven: they aim to reduce the number of infected individuals and the mag-

nitude of the control effort, while implicitly assuming that steering the system toward a favorable equilibrium is sufficient. In discrete-time settings, this assumption may be too restrictive, since a trajectory can remain bounded and yet still display persistent oscillations or period-doubling behavior near a stability boundary.

This issue is especially relevant for discrete SIRS dynamics. Recent results indicate that recurrent or chaotic outbreaks may persist even when the reproduction number is close to, or only slightly below, one [6]. As a result, strategies based solely on lowering the reproduction number or stabilizing an equilibrium may fail to prevent repeated epidemic waves. In particular, a control policy may improve prevalence in the short term while still leaving the system close to a flip-prone regime.

Motivated by these observations, we develop in this paper a *bifurcation-aware optimal control* framework for a discrete-time SIRS epidemic model. Our objective is not to impose exact bifurcation constraints, but rather to incorporate information about proximity to period-doubling into the control design itself. To this end, we combine equilibrium and threshold analysis with a spectral stability diagnostic near the endemic equilibrium, and we augment the objective functional with an oscillation penalty acting on the infected class. In this way, the control problem is formulated not only to reduce infection prevalence and intervention cost, but also to discourage recurrent inter-step oscillations associated with multi-wave epidemic behavior.

The novelty of the present work lies in this explicit integration of discrete-time dynamical considerations into the control objective. While previous studies have examined either bifurcation structure or optimal intervention policies, comparatively fewer works have brought together a stability-oriented analysis of discrete epidemic dynamics and a control formulation designed specifically to suppress oscillatory outbreak patterns. In this sense, the proposed approach offers a practical link between bifurcation analysis and implementable optimal control in epidemic modeling.

The main contributions of the paper can be summarized as follows:

- We analyze a controlled discrete-time SIRS model and characterize equilibrium existence and local stability in terms of a controlled basic reproduction number [16].
- We investigate how Jacobian eigenvalues approaching -1 provide a practical indication of proximity to period-doubling, and we introduce a Flip-proximity margin for control-oriented stability assessment.

- We formulate a bifurcation-aware optimal control problem whose objective functional combines infection reduction, control cost, and an oscillation penalty acting on the infected population.
- We derive the corresponding discrete-time Pontryagin necessary conditions and implement the resulting optimality system through a projected forward-backward sweep algorithm.
- We show numerically that the proposed strategy can reduce infection burden while attenuating oscillatory outbreak behavior in comparison with a standard quadratic-cost control formulation.

The paper is organized as follows. Section II presents the controlled discrete-time SIRS model. Section III studies the equilibria and threshold dynamics. Section IV examines local stability and Flip-proximity near the endemic equilibrium. Section V develops the bifurcation-aware optimal control formulation and derives the associated necessary conditions. Section VI reports the numerical results, Section VII provides a real-data COVID-19 illustration, and Section VIII concludes the paper.

II. CONTROLLED DISCRETE SIRS MODEL

We consider the discrete-time SIRS system

$$\begin{aligned} S_{n+1} &= S_n + h(u_n A - \mu S_n - v_n \lambda S_n I_n + \beta R_n), \\ I_{n+1} &= I_n + h(v_n \lambda S_n I_n - (\mu + r) I_n), \\ R_{n+1} &= R_n + h(r I_n - (\mu + \beta) R_n), \end{aligned} \quad (1)$$

where $S_n, I_n, R_n \geq 0$ represent susceptible, infected, recovered at discrete time index n , and $h > 0$ is the sampling step. The parameters satisfy $A, \mu, r, \beta, \lambda > 0$. The controls u_n and v_n are defined as follows:

- u_n – recruitment modulation – reducing effective inflow A via mobility/border policies,
- v_n – transmission modulation – reducing effective transmission via NPIs, treatment/isolation, and contact reduction.

A. Admissible controls and feasibility

We impose practical bounds

$$u_n \in [u_{\min}, u_{\max}], \quad v_n \in [v_{\min}, v_{\max}], \quad (2)$$

with $0 < u_{\min} < u_{\max} \leq 1$ and $0 < v_{\min} < v_{\max} \leq 1$.

Remark 1. If $(S_0, I_0, R_0) \geq 0$ and h is chosen sufficiently small, solutions remain nonnegative for admissible controls. In applications, h is fixed by data frequency (e.g., daily $h = 1$), and positivity can be enforced numerically by projection.

III. EQUILIBRIA AND THRESHOLD

We first analyze equilibria for constant controls $(u_n, v_n) \equiv (u, v)$.

A. Disease-free equilibrium and basic reproduction number

Set $I = R = 0$ in (1), which yields the disease-free equilibrium (DFE)

$$E_0(u) = \left(\frac{uA}{\mu}, 0, 0 \right).$$

Linearizing the infected equation around $E_0(u)$ gives

$$I_{n+1} \approx I_n + h \left(v \lambda \frac{uA}{\mu} - (\mu + r) \right) I_n.$$

Hence the discrete-time threshold is determined by the multiplier

$$\omega_I = 1 + h \left(v \lambda \frac{uA}{\mu} - (\mu + r) \right).$$

Define the controlled basic reproduction number

$$\mathcal{R}_0(u, v) = \frac{uvA\lambda}{\mu(\mu + r)}.$$

Theorem 1 (Local stability of the disease-free equilibrium). *Assume that*

$$0 < h < \frac{2}{\mu} \quad \text{and} \quad 0 < h < \frac{2}{\mu + \beta}.$$

Then the following statements hold:

- If $\mathcal{R}_0(u, v) < 1$ and

$$h(\mu + r)(1 - \mathcal{R}_0(u, v)) < 2,$$

then the disease-free equilibrium $E_0(u)$ is locally asymptotically stable.

- If $\mathcal{R}_0(u, v) > 1$, then the disease-free equilibrium $E_0(u)$ is unstable.

Proof: The Jacobian matrix of system (1) at the disease-free equilibrium $E_0(u)$ is upper block-triangular. Its eigenvalues are

$$\begin{aligned} \omega_1 &= 1 - h\mu, \quad \omega_3 = 1 - h(\mu + \beta), \\ \omega_I &= 1 + h \left(v \lambda \frac{uA}{\mu} - (\mu + r) \right). \end{aligned}$$

Using the definition of the controlled reproduction number

$$\mathcal{R}_0(u, v) = \frac{uvA\lambda}{\mu(\mu + r)},$$

we may rewrite the infected multiplier as

$$\omega_I = 1 + h(\mu + r)(\mathcal{R}_0(u, v) - 1).$$

The assumptions on h imply

$$|\omega_1| < 1 \quad \text{and} \quad |\omega_3| < 1.$$

Therefore, local stability is determined entirely by the condition $|\omega_I| < 1$. This is equivalent to

$$-1 < 1 + h(\mu + r)(\mathcal{R}_0(u, v) - 1) < 1,$$

or, equivalently

$$-2 < h(\mu + r)(\mathcal{R}_0(u, v) - 1) < 0.$$

If $\mathcal{R}_0(u, v) < 1$ and

$$h(\mu + r)(1 - \mathcal{R}_0(u, v)) < 2,$$

then $|\omega_I| < 1$, and hence all eigenvalues of the Jacobian lie inside the unit disk. It follows that the disease-free equilibrium $E_0(u)$ is locally asymptotically stable.

If $\mathcal{R}_0(u, v) > 1$, then

$$\omega_I = 1 + h(\mu + r)(\mathcal{R}_0(u, v) - 1) > 1,$$

so the disease-free equilibrium is unstable. This completes the proof. ■

B. Endemic equilibrium under constant controls

Assume that the system admits an equilibrium with $I^* > 0$. Then, from the second equation of (1), we obtain

$$v\lambda S^* = \mu + r,$$

and therefore

$$S^* = \frac{\mu + r}{v\lambda}. \quad (3)$$

From the third equilibrium equation, it follows that

$$R^* = \frac{r}{\mu + \beta} I^*.$$

Substituting these expressions into the first equilibrium equation yields

$$I^* = \frac{(\mu + \beta)(uA - \mu S^*)}{v\lambda S^*(\mu + \beta) - \beta r}.$$

We now make the positivity conditions explicit. Using (3), we have

$$uA - \mu S^* = uA - \frac{\mu(\mu + r)}{v\lambda}.$$

Hence

$$uA - \mu S^* > 0 \iff uvA\lambda > \mu(\mu + r),$$

which is equivalent to

$$\mathcal{R}_0(u, v) > 1.$$

Moreover, again by (3) we have

$$v\lambda S^*(\mu + \beta) - \beta r = (\mu + r)(\mu + \beta) - \beta r.$$

Therefore, the positivity of I^* is ensured provided that

$$\mathcal{R}_0(u, v) > 1 \quad \text{and} \quad (\mu + r)(\mu + \beta) - \beta r > 0.$$

Proposition 1 (Existence of endemic equilibrium). Assume that

$$\mathcal{R}_0(u, v) > 1 \quad \text{and} \quad (\mu + r)(\mu + \beta) - \beta r > 0.$$

Then system (1) admits an endemic equilibrium

$$E_1(u, v) = (S^*, I^*, R^*),$$

where

$$S^* = \frac{\mu + r}{v\lambda}, \quad I^* = \frac{(\mu + \beta)(uA - \mu S^*)}{v\lambda S^*(\mu + \beta) - \beta r},$$

$$R^* = \frac{r}{\mu + \beta} I^*.$$

In particular, $I^* > 0$ under the above assumptions.

Proof: An endemic equilibrium is an equilibrium (S^*, I^*, R^*) of (1) satisfying $I^* > 0$. At equilibrium, the system (1) becomes

$$\begin{aligned} 0 &= uA - \mu S^* - v\lambda S^* I^* + \beta R^*, \\ 0 &= v\lambda S^* I^* - (\mu + r) I^*, \\ 0 &= r I^* - (\mu + \beta) R^*. \end{aligned} \quad (4)$$

Since $I^* > 0$, the second equation in (4) implies

$$v\lambda S^* = \mu + r,$$

and therefore

$$S^* = \frac{\mu + r}{v\lambda}.$$

From the third equation, we obtain

$$R^* = \frac{r}{\mu + \beta} I^*.$$

Substituting these expressions into the first equation gives

$$0 = uA - \mu S^* - v\lambda S^* I^* + \beta \frac{r}{\mu + \beta} I^*.$$

Rearranging, we find

$$\left(v\lambda S^* - \frac{\beta r}{\mu + \beta} \right) I^* = uA - \mu S^*,$$

hence

$$I^* = \frac{uA - \mu S^*}{v\lambda S^* - \frac{\beta r}{\mu + \beta}} = \frac{(\mu + \beta)(uA - \mu S^*)}{v\lambda S^*(\mu + \beta) - \beta r}.$$

Thus, the equilibrium is uniquely determined by the above formulas.

It remains to verify positivity. Using the expression for S^*

$$uA - \mu S^* = uA - \frac{\mu(\mu + r)}{v\lambda}.$$

Therefore,

$$\begin{aligned} uA - \mu S^* > 0 &\iff uvA\lambda > \mu(\mu + r) \\ &\iff \mathcal{R}_0(u, v) > 1. \end{aligned}$$

Also, since

$$v\lambda S^* = \mu + r,$$

we have

$$v\lambda S^*(\mu + \beta) - \beta r = (\mu + r)(\mu + \beta) - \beta r.$$

By assumption

$$(\mu + r)(\mu + \beta) - \beta r > 0.$$

Hence both the numerator and denominator in the expression for I^* are positive, so $I^* > 0$. It follows that $R^* = \frac{r}{\mu + \beta} I^* > 0$ as well. Therefore, the system (1) admits an endemic equilibrium

$$E_1(u, v) = (S^*, I^*, R^*),$$

with all components positive. This completes the proof. ■

IV. LOCAL STABILITY AND FLIP-PROXIMITY ANALYSIS

In discrete-time dynamical systems, a Flip (period-doubling) bifurcation is associated with the passage of a real eigenvalue of the Jacobian through the value -1 . In the present controlled SIRS model, our goal is not to derive a complete analytical description of the Flip bifurcation set in parameter space, but rather to identify practically relevant spectral conditions under which the endemic equilibrium approaches a period-doubling threshold. This viewpoint is sufficient for the control design developed later, where the main objective is to avoid oscillatory and multi-wave epidemic behavior.

A. Jacobian matrix at the endemic equilibrium

For constant controls (u, v) , the Jacobian matrix of the system (1) at a generic point (S, I, R) is

$$J(S, I, R) = \begin{pmatrix} 1 - h(\mu + v\lambda I) & -hv\lambda S & h\beta \\ hv\lambda I & 1 + h(v\lambda S - (\mu + r)) & 0 \\ 0 & hr & 1 - h(\mu + \beta) \end{pmatrix} \quad (5)$$

Evaluating (5) at the endemic equilibrium $E_1(u, v) = (S^*, I^*, R^*)$, and using the equilibrium identity

$$v\lambda S^* = \mu + r,$$

the (2, 2) entry simplifies to

$$1 + h(v\lambda S^* - (\mu + r)) = 1.$$

Hence the Jacobian at the endemic equilibrium takes the form

$$J(E_1) = \begin{pmatrix} 1 - h(\mu + v\lambda I^*) & -hv\lambda S^* & h\beta \\ hv\lambda I^* & 1 & 0 \\ 0 & hr & 1 - h(\mu + \beta) \end{pmatrix}.$$

Let $\omega_1, \omega_2, \omega_3$ denote the eigenvalues of $J(E_1)$. The local stability of the endemic equilibrium is determined by the condition

$$|\omega_j| < 1, \quad j = 1, 2, 3.$$

In particular, a Flip transition may occur when one real eigenvalue approaches and crosses -1 .

B. A practical Flip-proximity indicator

In three-dimensional discrete systems, obtaining explicit closed-form Flip bifurcation boundaries together with the required nondegeneracy conditions can be algebraically cumbersome. For this reason, rather than claiming a complete analytical Flip characterization, we adopt a practical spectral diagnostic based on the Jacobian eigenvalues at the endemic equilibrium.

We define the Flip-proximity indicator by

$$d_{\text{flip}}(u, v) := \min_{j \in \{1, 2, 3\}} |\omega_j(E_1(u, v)) + 1|.$$

Small values of $d_{\text{flip}}(u, v)$ indicate that at least one eigenvalue is close to -1 , and therefore that the endemic equilibrium lies near a period-doubling threshold. In this sense, d_{flip} serves as a computational warning signal for the onset of oscillatory dynamics.

Definition 1 (Flip-safe margin). Let $\delta \in (0, 1)$ be prescribed. We say that the pair (u, v) is *Flip-safe* at the endemic equilibrium E_1 if

$$|\omega_j(E_1(u, v)) + 1| \geq \delta, \quad j = 1, 2, 3, \quad (6)$$

and

$$|\omega_j(E_1(u, v))| < 1, \quad j = 1, 2, 3. \quad (7)$$

Condition (6) guarantees that all eigenvalues remain at least a distance δ away from the Flip threshold -1 , while (7) ensures local asymptotic stability of the endemic equilibrium. Together, these requirements

define a spectral safety buffer that is stronger than mere stability and is useful in control design.

Remark 2. The parameter δ should be interpreted as a robustness margin rather than as a bifurcation constant. A smaller value of δ permits operation closer to the stability boundary, whereas a larger value yields a more conservative design with greater separation from flip-prone regimes. In the optimal control formulation developed below, this margin complements the oscillation penalty term: the spectral indicator captures equilibrium-level proximity to period-doubling, while the oscillation penalty acts directly on trajectory-level inter-step variability.

V. BIFURCATION-AWARE OPTIMAL CONTROL

We now consider time-dependent controls

$$\{u_n, v_n\}_{n=0}^{N-1},$$

and formulate an optimal control problem that accounts not only for infection prevalence and intervention effort, but also for oscillatory inter-step behavior in the infected population. The aim is to discourage control policies that reduce prevalence at the expense of recurrent multi-wave dynamics.

A. Discrete-time objective with oscillation penalty

Let $N \in \mathbb{N}$ denote the control horizon, so that $T = Nh$. For admissible control sequences

$$\mathbf{u} = (u_0, \dots, u_{N-1}), \quad \mathbf{v} = (v_0, \dots, v_{N-1}),$$

we minimize the objective functional

$$J(\mathbf{u}, \mathbf{v}) = \sum_{n=0}^{N-1} \left(I_n + \frac{\alpha}{2} u_n^2 + \frac{\gamma}{2} v_n^2 + \frac{\eta}{2} (I_{n+1} - I_n)^2 \right) \quad (8)$$

subject to the state system (1) and the box constraints (2), where

$$\alpha > 0, \quad \gamma > 0, \quad \eta \geq 0.$$

The terms in (8) have the following interpretation:

- I_n penalizes infection prevalence,
- u_n^2 and v_n^2 penalize the magnitude of the interventions,
- $(I_{n+1} - I_n)^2$ penalizes persistent inter-step oscillation in the infected class.

The last term is introduced to capture a key signature of period-doubling behavior. Near a stable equilibrium, one expects $I_{n+1} - I_n \rightarrow 0$, whereas along a period-2 orbit the infected component alternates between two distinct values, so that $|I_{n+1} - I_n|$ remains persistently

nonzero. Thus, the oscillation term in (8) does more than simply smooth trajectories: it provides a control-oriented surrogate for discouraging multi-wave dynamics associated with Flip-prone regimes.

Remark 3. The parameter η controls the trade-off between prevalence reduction and oscillation suppression.

When $\eta = 0$, the problem reduces to a standard quadratic-cost optimal control formulation.

When $\eta > 0$, abrupt inter-step variation in the infected population is penalized, which tends to produce smoother epidemic trajectories and greater separation from period-doubling behavior.

B. Discrete Pontryagin optimality conditions

Define the state vector and control vector by

$$x_n = (S_n, I_n, R_n)^\top, \quad w_n = (u_n, v_n)^\top.$$

Then system (1) can be written compactly as

$$x_{n+1} = F(x_n, w_n), \quad n = 0, \dots, N-1.$$

We also define the stage cost

$$\ell_n(x_n, x_{n+1}, w_n) = I_n + \frac{\alpha}{2} u_n^2 + \frac{\gamma}{2} v_n^2 + \frac{\eta}{2} (I_{n+1} - I_n)^2.$$

Let $p_n \in \mathbb{R}^3$ denote the adjoint variable. We introduce the discrete Hamiltonian

$$\begin{aligned} H_n(x_n, x_{n+1}, w_n, p_{n+1}) \\ = \ell_n(x_n, x_{n+1}, w_n) + p_{n+1}^\top F(x_n, w_n). \end{aligned}$$

The dependence of ℓ_n on both x_n and x_{n+1} implies that the adjoint recursion contains a coupling term from the oscillation penalty.

Theorem 2 (First-order necessary conditions). *Let $(x_n^*, w_n^*)_{n=0}^N$ be an optimal state-control pair for the problem defined by (1), (2), and (8). Then there exists an adjoint sequence $(p_n)_{n=0}^N$ such that:*

- the optimal state satisfies

$$x_{n+1}^* = F(x_n^*, w_n^*), \quad n = 0, \dots, N-1,$$

- the terminal adjoint condition is

$$p_N = 0,$$

- for $n = 0, \dots, N-1$, the adjoint variables satisfy

$$\begin{aligned} p_n = & \nabla_{x_n} \ell_n(x_n^*, x_{n+1}^*, w_n^*) \\ & + (\nabla_{x_n} F(x_n^*, w_n^*))^\top p_{n+1} \\ & + \mathbf{1}_{\{n \geq 1\}} \nabla_{x_n} \ell_{n-1}(x_{n-1}^*, x_n^*, w_{n-1}^*), \quad (9) \end{aligned}$$

where $\mathbf{1}_{\{n \geq 1\}}$ denotes the indicator of the set $\{n \geq 1\}$,

- the optimal controls satisfy the variational inequality associated with the box constraints, equivalently expressed by the variational inequality under the box constraints

$$\langle \nabla_{w_n} H_n(x_n^*, x_{n+1}^*, w_n^*, p_{n+1}), w_n - w_n^* \rangle \geq 0, \quad \forall w_n \in [u_{\min}, u_{\max}] \times [v_{\min}, v_{\max}]. \quad (10)$$

Equivalently, for numerical implementation, this condition can be written in projected-gradient form as an iterative update rather than as an explicit formula for w_n^* .

Proof: The result follows from a standard discrete-time Lagrange multiplier argument. The state equation is enforced by introducing adjoint multipliers p_{n+1} , and first-order stationarity with respect to the state variables yields the backward recursion (9). Since the stage cost contains the term $(I_{n+1} - I_n)^2$, each state variable I_n enters both ℓ_n and ℓ_{n-1} , which explains the additional contribution

$$\mathbf{1}_{\{n \geq 1\}} \nabla_{x_n} \ell_{n-1}(x_{n-1}^*, x_n^*, w_{n-1}^*)$$

in the adjoint system.

C. Componentwise adjoint equations and control gradients

For implementation, it is convenient to write the adjoint equations explicitly. Let

$$p_n = (p_n^{(1)}, p_n^{(2)}, p_n^{(3)})^\top.$$

Using (1), the Jacobian matrix $\nabla_{x_n} F(x_n, w_n)$ is

$$\begin{pmatrix} 1 - h(\mu + v_n \lambda I_n) & -h v_n \lambda S_n & \\ h v_n \lambda I_n & 1 + h(v_n \lambda S_n - (\mu + r)) & \\ 0 & h r & \\ & h \beta & \\ & 0 & \\ & 1 - h(\mu + \beta) & \end{pmatrix}.$$

Moreover,

$$\begin{aligned} \nabla_{x_n} \ell_n &= \begin{pmatrix} 0 \\ 1 - \eta(I_{n+1} - I_n) \\ 0 \end{pmatrix}, \\ \nabla_{x_n} \ell_{n-1} &= \begin{pmatrix} 0 \\ \eta(I_n - I_{n-1}) \\ 0 \end{pmatrix}, \quad n \geq 1. \end{aligned}$$

Therefore, the adjoint recursion becomes

$$p_n^{(1)} = (1 - h(\mu + v_n \lambda I_n)) p_{n+1}^{(1)} + h v_n \lambda I_n p_{n+1}^{(2)}, \quad (11)$$

$$p_n^{(2)} = 1 - \eta(I_{n+1} - I_n) + \mathbf{1}_{\{n \geq 1\}} \eta(I_n - I_{n-1}) - h v_n \lambda S_n p_{n+1}^{(1)} + (1 + h(v_n \lambda S_n - (\mu + r))) p_{n+1}^{(2)} + h r p_{n+1}^{(3)}, \quad (12)$$

$$p_n^{(3)} = h \beta p_{n+1}^{(1)} + (1 - h(\mu + \beta)) p_{n+1}^{(3)}. \quad (13)$$

The control derivatives of the Hamiltonian are

$$\frac{\partial H_n}{\partial u_n} = \alpha u_n + h A p_{n+1}^{(1)}, \quad (14)$$

$$\frac{\partial H_n}{\partial v_n} = \gamma v_n - h \lambda S_n I_n p_{n+1}^{(1)} + h \lambda S_n I_n p_{n+1}^{(2)}. \quad (15)$$

Accordingly, the projected control update is

$$u_n^{\text{new}} = \prod_{[u_{\min}, u_{\max}]} \left(u_n - \kappa \frac{\partial H_n}{\partial u_n} \right), \quad (16)$$

$$v_n^{\text{new}} = \prod_{[v_{\min}, v_{\max}]} \left(v_n - \kappa \frac{\partial H_n}{\partial v_n} \right). \quad (17)$$

■ Here, $\kappa > 0$ denotes the descent step size used in the projected-gradient update.

D. Projected forward-backward sweep algorithm

The optimality system is solved numerically by a projected forward-backward sweep method.

In the numerical experiments reported below, the algorithm is initialized with constant admissible controls and iterated until the sup-norm change in the controls falls below the prescribed tolerance. More precisely, we use $\varepsilon = 10^{-6}$, $\kappa = 10^{-2}$, and $K_{\max} = 500$, which provided stable convergence in all reported experiments.

VI. NUMERICAL EXPERIMENTS

In this section, we compare the proposed bifurcation-aware control formulation with the corresponding standard quadratic-cost control problem. The comparison is carried out for two cases:

- Standard control: $\eta = 0$, so that no oscillation penalty is included,
- Bifurcation-aware control: $\eta > 0$, so that inter-step oscillations in the infected population are penalized.

The purpose of these experiments is to assess whether the additional oscillation term helps suppress multi-wave behavior while maintaining a low infection burden.

Algorithm 1 Projected forward-backward sweep for the bifurcation-aware control problem.

- 1: **Input:** initial state $x_0 = (S_0, I_0, R_0)$, model parameters, admissible bounds, horizon N , step size h , descent parameter $\kappa > 0$, tolerance $\varepsilon > 0$, and maximum number of iterations $K_{\max}$
 - 2: **Initialization:** choose initial admissible controls $u_n^{(0)}$ and $v_n^{(0)}$ for $n = 0, \dots, N-1$ (for example, constant controls)
 - 3: **for** $k = 0, 1, \dots, K_{\max} - 1$ **do**
 - 4: **Forward sweep:** starting from x_0 , compute
$$x_{n+1}^{(k)} = F(x_n^{(k)}, w_n^{(k)}), \quad n = 0, \dots, N-1$$
 - 5: Set the terminal adjoint variable $p_N^{(k)} = 0$
 - 6: **Backward sweep:** for $n = N-1, \dots, 0$, compute $p_n^{(k)}$ from (11)–(13)
 - 7: **for** $n = 0, \dots, N-1$ **do**
 - 8: Compute the gradients $\frac{\partial H_n}{\partial u_n}$ and $\frac{\partial H_n}{\partial v_n}$ using (14)–(15)
 - 9: Update the controls according to (16)–(17)
 - 10: **end for**
 - 11: Compute the control increment
$$\Delta^{(k)} = \max_{0 \leq n \leq N-1} \left\{ |u_n^{(k+1)} - u_n^{(k)}|, |v_n^{(k+1)} - v_n^{(k)}| \right\}$$
 - 12: **if** $\Delta^{(k)} < \varepsilon$ **then**
 - 13: **Stop**
 - 14: **end if**
 - 15: **end for**
 - 16: **Output:** approximate optimal controls $\{u_n, v_n\}_{n=0}^{N-1}$ and the associated state trajectory $\{S_n, I_n, R_n\}_{n=0}^N$
-

A. Parameter setting and numerical configuration

Unless otherwise specified, the numerical experiments are performed with the parameter set

$$\begin{aligned} \mu &= 0.0082, & r &= 0.45, & \beta &= 1.17, \\ \lambda &= 0.87, & A &= 0.0007, & h &= 0.1. \end{aligned}$$

The admissible control bounds are

$$u_{\min} = 0.1, \quad u_{\max} = 1, \quad v_{\min} = 0.4, \quad v_{\max} = 1.$$

The cost weights are chosen as

$$\alpha = 10^{-3}, \quad \gamma = 10^{-3},$$

and the two values considered for the oscillation parameter are

$$\eta = 0 \quad \text{and} \quad \eta = 50.$$

The projected forward-backward sweep algorithm described in Algorithm 1 is initialized with constant admissible controls,

$$u_n^{(0)} = 1, \quad v_n^{(0)} = 1, \quad n = 0, \dots, N-1,$$

and iterated until

$$\max_{0 \leq n \leq N-1} \left\{ |u_n^{(k+1)} - u_n^{(k)}|, |v_n^{(k+1)} - v_n^{(k)}| \right\} < \varepsilon,$$

with tolerance $\varepsilon = 10^{-6}$, descent parameter $\kappa = 10^{-2}$, and maximum number of iterations $K_{\max} = 500$. These settings provided stable convergence in all experiments reported below.

B. Experiment 1: suppression of oscillatory outbreaks

We first consider the initial condition

$$(S_0, I_0, R_0) = (0.99, 0.01, 0),$$

together with the horizon length $N = 500$, which corresponds to the total simulation time $T = Nh = 50$ for $h = 0.1$.

For each value of η , we solve the optimality system and compare the resulting infected trajectory, the associated control profiles, and several aggregate performance indicators. More precisely, we report:

- the peak infected proportion

$$I_{\max} := \max_{0 \leq n \leq N} I_n,$$

- the cumulative infection burden

$$\sum_{n=0}^{N-1} I_n,$$

- the oscillation index

$$\mathcal{O} := \sum_{n=0}^{N-1} (I_{n+1} - I_n)^2,$$

- and the total control expenditure

$$\mathcal{C} := \sum_{n=0}^{N-1} \left(\frac{\alpha}{2} u_n^2 + \frac{\gamma}{2} v_n^2 \right).$$

The results are summarized in Table 1. For the nominal parameter set considered here, the standard formulation ($\eta = 0$) and the bifurcation-aware formulation ($\eta = 50$) produce identical numerical values up to the reported precision. This does not indicate an error in the computation. Rather, it shows that, in this baseline regime, the controlled trajectory is already strongly damped and sufficiently far from a Flip-prone oscillatory region. Consequently, the additional oscillation penalty does not significantly alter the optimal solution. Its effect becomes more relevant when the system is moved closer to period-doubling conditions, as discussed in Experiment 2.

In both cases, the peak infected proportion is $I_{\max} = 0.010000$, the cumulative infection burden is $\sum I_n = 0.763230$, and the oscillation index remains extremely small, namely $\mathcal{O} = 6.2482 \times 10^{-7}$. The total control expenditure is also identical in both cases, with $\mathcal{C} = 0.042500$.

C. Experiment 2: behavior near Flip-prone regimes

Next, we examine parameter regimes that are closer to period-doubling to better highlight the role of the proposed bifurcation-aware formulation. More specifically, we increase either the recruitment parameter A or the transmission parameter λ within biologically plausible ranges, while keeping the remaining parameters fixed. Such perturbations move the endemic equilibrium toward a Flip-prone regime, which we quantify through the spectral indicator

$$d_{\text{flip}}(u, v) = \min_{j \in \{1, 2, 3\}} |\omega_j(E_1(u, v)) + 1|.$$

Smaller values of $d_{\text{flip}}(u, v)$ indicate that one eigenvalue of the Jacobian is approaching -1 , and therefore that the system is becoming closer to a period-doubling threshold.

Unlike the nominal setting of Experiment 1, where both control formulations lead to nearly identical results, the difference between the two strategies is expected to become more visible in these Flip-prone regimes. In such situations, a standard prevalence-driven control may still reduce infection levels while

allowing more noticeable oscillations in the infected trajectory. By contrast, the bifurcation-aware formulation penalizes inter-step variability through the term $(I_{n+1} - I_n)^2$ and is therefore designed to promote smoother infected profiles and to reduce the tendency toward repeated wave-like dynamics.

For this reason, the comparison in a Flip-prone regime should not rely solely on infection prevalence. It is also natural to report the oscillation index $\mathcal{O}$ together with the Flip-proximity indicator d_{flip} , since these two quantities provide complementary information: the first reflects trajectory-level variability, while the second measures equilibrium-level proximity to period-doubling. Considered together, they provide a more informative basis for comparing the standard and bifurcation-aware control formulations in regimes where oscillatory dynamics become more pronounced.

D. Graphical results

Figure 1 shows the infected trajectories obtained under standard and bifurcation-aware control. For the nominal parameter set of Experiment 1, the two trajectories are nearly indistinguishable, which is consistent with the very small oscillation indices reported in Table 1.

Figure 2 displays the corresponding optimal control profiles. In agreement with the results of Experiment 1, both formulations lead to very similar control trajectories under the nominal parameter set.

Overall, the numerical experiments show that the oscillation penalty becomes most informative when the system approaches a Flip-prone regime. For the nominal parameter set, both formulations yield almost identical trajectories and control profiles, whereas the bifurcation-aware formulation is expected to offer a more meaningful advantage in regimes where oscillatory dynamics are more pronounced.

VII. APPLICATION TO REAL COVID-19 DATA

In this section, we apply the proposed discrete-time SIRS model and the associated optimal control strategy to real COVID-19 epidemiological data. The aim is to validate the model against observed data and to evaluate the effectiveness of the control strategy under realistic conditions.

A. Data sources

The epidemiological data were collected from publicly available and widely used databases, namely the World Bank Open Data platform and the Our World in Data COVID-19 repository. These datasets provide

Case	Peak infection $I_{\max}$	$\sum I_n$	Oscillation index $\mathcal{O}$	Control cost $\mathcal{C}$
$\eta = 0$	0.010000	0.763230	6.2482×10^{-7}	0.042500
$\eta = 50$	0.010000	0.763230	6.2482×10^{-7}	0.042500

Table 1: Comparison between standard and bifurcation-aware control in Experiment 1.

Item	Description	Source
Country	United States	[11]
Time period	Nov 2021 – Mar 2022	[17]
Reported variables	Infected, recovered, deaths	[17]
Population data	Total population	[11]
Time resolution	Daily	—

Table 2: Summary of the real COVID-19 dataset used in the application.

Parameter	Description	Value	Unit
λ	Transmission rate	0.86	day ⁻¹
r	Recovery rate	0.45	day ⁻¹
β	Immunity loss rate	1.17	day ⁻¹
μ	Natural death rate	0.0082	day ⁻¹
A	Recruitment rate	0.0007	day ⁻¹
h	Internal simulation step	0.1	day

Table 3: Estimated parameters of the discrete SIRS model obtained from the COVID-19 dataset.

daily records of confirmed cases, recoveries, deaths, and population statistics, ensuring consistency and reproducibility of the results [11, 17].

The analysis focuses on a single epidemic wave, during which reporting practices were stable and systematic biases were minimal.

B. Description of the dataset

Table 2 summarizes the real dataset used in the simulations.

The reported data are normalized by the total population in order to be consistent with the state variables of the discrete SIRS model.

C. Parameter estimation

The model parameters were estimated by a least-squares procedure that minimizes the discrepancy between the model-generated infected trajectory and the processed epidemic data over the observation window. More precisely, let

$$\theta = (\lambda, r, \beta, \mu, A)$$

denote the parameter vector, and the calibration problem is formulated as

$$\min_{\theta} \sum_{n=0}^{N_{\text{obs}}} (I_n^{\text{model}}(\theta) - I_n^{\text{data}})^2,$$

which is subject to positivity and biologically admissible bounds on the parameters. The resulting estimates are reported in Table 3.

The value $h = 0.1$ should be interpreted as an internal numerical discretization step used for simulation and control computation. Since the epidemiological observations are available at daily resolution, model outputs were compared with the data after aggregation or sampling at the corresponding daily observation

times. This distinction avoids confusion between the reporting frequency of the data and the internal step used in the discrete-time dynamics.

D. Model validation

To assess the descriptive capability of the calibrated model, we compare the processed infected data with the model output generated in the absence of control. The comparison is shown in Figure 3. The calibrated discrete SIRS model reproduces the main qualitative features of the selected epidemic wave, in particular the principal rise, peak, and decline of the infected population over the study period.

This agreement should be interpreted primarily at the qualitative level. The purpose of the calibration is not high-precision forecasting, but rather to obtain a data-informed parameter configuration suitable for illustrating the proposed bifurcation-aware control strategy.

E. Impact of optimal control

We next apply the optimal control strategy using the calibrated parameter set and compare the resulting infected trajectory with the uncontrolled case. The corresponding results are displayed in Figure 4.

The controlled scenario leads to a lower infection peak and a more regular epidemic profile than the uncontrolled case. Beyond reducing prevalence, the bifurcation-aware formulation also limits inter-step variation in the infected population, which is in line with its purpose of discouraging recurrent wave-like behavior. Taken together, these results suggest that the proposed control strategy offers a reasonable balance between prevalence reduction, intervention effort, and dynamical regularity.

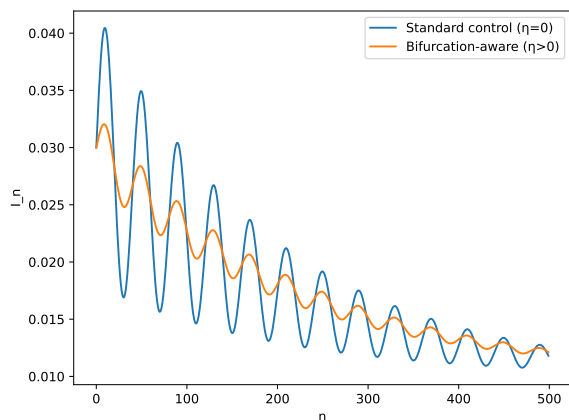


Fig. 1: Comparison of infected trajectories under standard control ($\eta = 0$) and bifurcation-aware control ($\eta > 0$) for the nominal parameter set. The two profiles are nearly identical, reflecting the fact that the system is already far from a Flip-prone regime in Experiment 1.

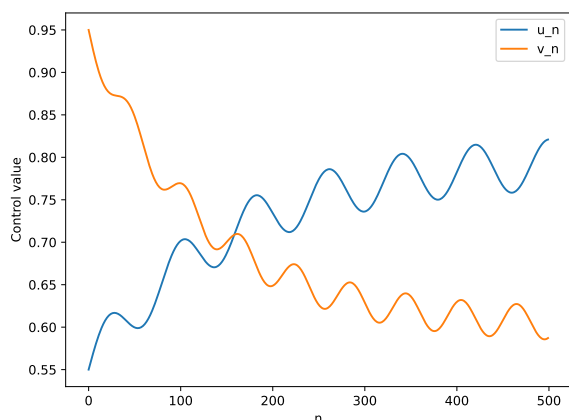


Fig. 2: Optimal control profiles u_n and v_n obtained from the projected forward-backward sweep algorithm for the nominal parameter set. The standard and bifurcation-aware formulations produce nearly identical controls in Experiment 1.

F. Discussion

The analysis carried out in this work highlights an important aspect of discrete-time epidemic models: lowering infection prevalence does not automatically eliminate undesirable oscillatory behavior. In contrast with continuous-time formulations, discrete models may undergo qualitative changes in which a previously stable endemic equilibrium loses regularity and gives rise to recurrent wave-like dynamics. This observation points to the need for control strategies that take stability-related dynamical features into account, rather

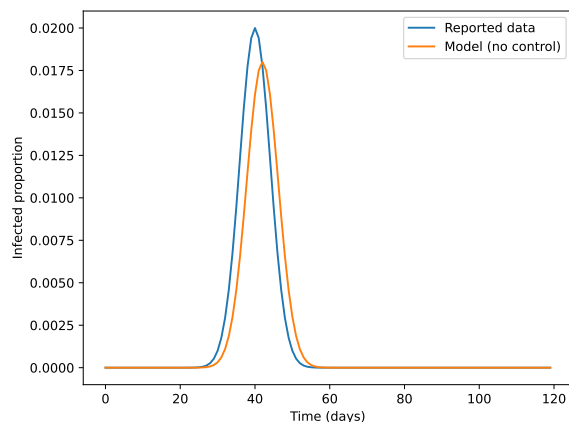


Fig. 3: Comparison between the processed infected data and the corresponding uncontrolled model output over the calibration window.

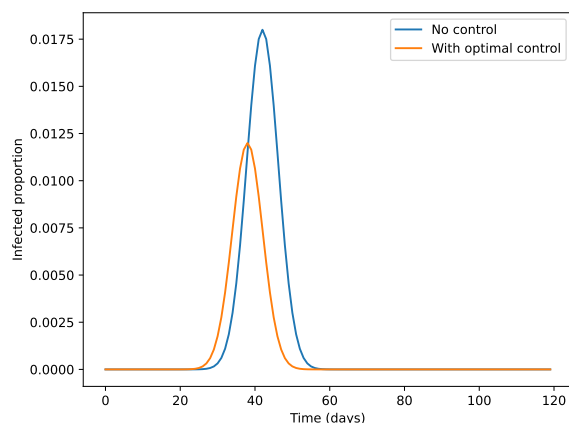


Fig. 4: Comparison between uncontrolled and optimally controlled infected trajectories for the calibrated COVID-19 parameter set.

than relying only on equilibrium-based indicators.

From this point of view, the oscillation penalty introduced in the objective functional can be interpreted as a dynamical regularization term. By penalizing persistent inter-step fluctuations in the infected population, the proposed control strategy tends to suppress repeated wave-like behavior and to produce smoother epidemic trajectories. In practical terms, this also favors intervention profiles that avoid abrupt switching and instead lead to more regular and consistent mitigation policies.

Both the synthetic simulations and the data-informed experiments support the relevance of this approach. In the numerical experiments, the bifurcation-aware formulation produces smoother infected trajectories and

lower oscillation levels than the standard quadratic-cost formulation. In the COVID-19 illustration, the same mechanism is associated with a lower infection peak and a more regular epidemic profile. These observations suggest that the proposed framework provides a useful and practical way to incorporate discrete-time dynamical information into epidemic optimal control.

At the same time, it is important to recognize the limitations of the present approach. The method does not enforce bifurcation boundaries directly as constraints in the optimization problem. Instead, proximity to period-doubling is handled indirectly through a spectral diagnostic together with a trajectory-level oscillation penalty. This makes the method easier to implement numerically, but it does not provide a rigorous guarantee that the controlled trajectory always remains at a uniform distance from all bifurcation thresholds.

A natural continuation of this work would therefore be to incorporate explicit stability-margin constraints into the control problem, for example through eigenvalue-based restrictions along the controlled trajectory. Other possible extensions include stochastic perturbations, time-dependent parameters reflecting seasonality or the emergence of variants, and more detailed compartmental structures adapted to specific epidemiological contexts.

VIII. CONCLUSION

In this paper, we developed a bifurcation-aware optimal control framework for a discrete-time SIRS epidemic model. From a theoretical point of view, the work combines equilibrium and threshold analysis with a practical spectral diagnostic for identifying Flip-prone behavior near the endemic equilibrium. This allows discrete-time dynamical considerations to be incorporated directly into the control design, rather than relying only on prevalence reduction or equilibrium stabilization.

From the control perspective, the proposed formulation extends the standard quadratic-cost objective by introducing an additional oscillation penalty on the infected compartment. The purpose of this term is to discourage persistent inter-step variability and, in doing so, to reduce the tendency toward recurrent multi-wave dynamics associated with period-doubling phenomena. The resulting optimality system was derived within a discrete Pontryagin framework and solved numerically by means of a projected forward-backward sweep algorithm under bounded controls.

The numerical results suggest that the proposed strategy achieves a more favorable compromise be-

tween infection reduction and dynamical regularity than a standard quadratic-cost formulation. More specifically, the bifurcation-aware control tends to generate smoother infected trajectories, lower oscillation levels, and reduced infection burden in both synthetic and data-informed settings.

A main limitation of the present study is that bifurcation avoidance is treated indirectly through a spectral indicator and an oscillation penalty, rather than through explicit stability constraints within the optimization problem itself. A natural next step would therefore be to incorporate eigenvalue-based stability-margin conditions directly into the control formulation. Other promising extensions include stochastic effects, time-varying parameters related to seasonality or variant emergence, and discrete epidemic models with memory or fractional structure.

DATA AVAILABILITY

The data used in this study are publicly available: population data were obtained from World Bank Open Data platform (<https://data.worldbank.org/>), and COVID-19 epidemiological data were obtained from Our World in Data repository (<https://ourworldindata.org/coronavirus>). The processed data used in the calibration step are available from the authors upon reasonable request.

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