

Optimal control metapopulation models for the West Africa Ebola epidemic and their nonstandard finite difference discretizations

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Abstract: The 2014-2016 Ebola Virus disease (EVD) outbreak came with an unprecedented challenge. It simultaneously arose in three different countries with considerable inbound and outbound travels. This led the World Health Organization (WHO) to suggest implementing exit screening of travelers at their borders. In addition to this measure, some countries accounted for entry screening and/or vaccination to curb the disease. In this paper, we consider two optimal controls for patch models in which other interventions supplement exit screening. The first problem is defined through the exit and entry screenings of travelers. The second involves the combination of exit screening and vaccination interventions. We prove the well-posedness of both optimal control problems. For each problem, we construct a new Nonstandard Finite Difference Forward-Backward-Sweep-Method, which preserves the dynamics of the continuous model. Numerical simulations that support the theory are provided. They

also suggest that vaccination alone is effective in reducing the disease burden.

Keywords: Ebola, exit and entry screenings, vaccination, optimal control problem, nonstandard finite difference schemes, forward-backward sweep method

I. INTRODUCTION

Since 1976, there have been about 30 Ebola Virus Disease (EVD) outbreaks, including the 2014-2016 West Africa EVD, which is the largest and deadliest. More than 28,000 cases were reported, of which 11,310 died from the disease [1,2]. This outbreak came with the unprecedented challenge of simultaneously occurring in three neighboring countries (*viz.* Guinea, Liberia, and Sierra Leone) with considerable inbound and outbound

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travels. It started in Guinea and later spread to Liberia and Sierra Leone due to migration [3,4].

Therefore, it is not surprising that the World Health Organization (WHO) declared on 8 August 2014 [5] a Public Health Emergency of International Concern (PHEIC) for the 2014-2016 Ebola outbreak and recommended the exit screening of travelers at international airports, seaports and major land crossings in the three most affected countries to prevent cross-border transmission of EVD [6, 7]. However, this preventive measure failed to confine the disease to the extent that some infected travelers arrived in Italy, Spain, the United Kingdom, USA, Mali, Senegal, and Nigeria [2, 8, 9], and triggered severe or mild outbreaks (secondary infections). Despite the timely and rapid response in Nigeria, the country registered at least 19 EVD cases [8]. The first EVD case who traveled from West Africa to Dallas had infected at least two health workers in the USA [2]. Thus, a need to look for measures that could help with better management and optimal control of EVD in the future.

A lot of mathematical models have been developed to understand the spread of EVD. In this regard, the work [10] provides a brief literature review to highlight: (a) the contributions of a large number of models that assessed the efficiency of strategies such as quarantine, isolation, vaccination, contact tracing and media coverage [11–15], (b) very few contributions were devoted to the influence of emigration/migration on the spatial spread of EVD [16–18], and (c) the lack from these works of considering the importance of screening travelers at exit borders, though exit and entry screenings were implemented for many other infectious diseases such as mosquitoes-borne ones and SARS [19, 20].

The authors in [10] filled these gaps by building and analyzing a metapopulation model with exit screening, quarantine, and isolation measures. They proved the usefulness of the exit screening measure to mitigate the number of EVD-infected cases, but suggested its combination with other measures, such as the entry screening for disease control improvement.

This motivates the implementation in the current work of two additional control measures, namely the entry screening measure, which was adopted by several countries during the 2014-2016 EVD outbreak [21–23], and the vaccination [25]. More precisely, we extend the work [10] by considering optimal control metapopulation models with entry-exit screening as control commands on the one hand and exit screening and vaccination on the other. The aim in both cases is to minimize the number of infected individ-

uals and the cost of control measures. We prove the existence and uniqueness of the optimal controls for these problems. Moreover, we design two Nonstandard Finite Difference Forward-Backward Sweep Methods (NSFD-FBSM) that are dynamically consistent with the continuous model. We generate numerical simulations that support the theory and show, in particular, the better performance of the nonstandard approach compared to the classical FBSM.

The rest of the paper is structured as follows. In Section II, we present a metapopulation model with exit-entry screening alongside its theoretical analysis with and without optimal entry screening. Section III deals with the optimal control problem which combines the exit screening and the vaccination. In each of these sections, the corresponding NSFD-FBSM is developed. In Section IV, numerical simulations are provided based on real data from the three most affected patches/countries by EVD, namely Guinea, Liberia, and Sierra Leone. Section V gathers concluding remarks, some recommendations, and planned future research work.

II. CONTROL THROUGH EXIT-ENTRY SCREENING

A. Model formulation

In this subsection, we implement exit and entry screenings to a metapopulation EVD model. Positively screened travelers are quarantined as individuals who are suspected of being infected with EVD [7]. In the quarantine compartment, they undergo some laboratory tests. Those found to be positive are sent to isolation, while the others can travel. Our model is built on the following main assumptions.

- 1) Only susceptible and latent individuals can travel between patches and travel at the same rate.
- 2) Only entry screening is optimal. The first implication of this assumption is that, unlike exit screening, which can lead to false positives (i.e. susceptible travelers can be positively screened as EVD-infected), the entry screening cannot, i.e. only latent individuals can be positively screened at the entry of patches. The motivation behind an optimal entry screening instead of exit screening is that the former measure serves a patch that implements it (as a result, states will be more keen to implement it effectively), while the latter serves the other patches. However, the relevance of optimal exit screening will be explored in the next section.
- 3) The isolation and quarantine compartments involved in this work are only for travelers stopped due to exit

Class	Description
S_i	Susceptible individuals in patch i who did not undergo screening or who were negatively screened.
E_i	Latent individuals in patch i .
I_i	EVD symptomatic cases in patch i .
D_i	Ebola-death cases in patch i who are not safely buried.
Q_i	Travelers positively screened at the exit borders and quarantined in patch i .
P_i	Isolated individuals in patch i who failed to travel due to a positive screening, followed by a positive diagnosis at the exit borders.
R_i	Individuals who recover from EVD in patch i .

Table 1: Variables of the model.

or entry screening. Specifically, we do not consider quarantine or isolation measures that follow contact tracing or case findings within the population. We refer the readers interested in such cases to [24,25]. In particular, the status of quarantined individuals is for travelers positively screened.

- 4) Positively diagnosed individuals who succumb to the disease are safely buried.
- 5) All symptomatic EVD cases that die outside isolation centers, regardless of the cause, are infectious.

We consider mutually exclusive model variables $S_i, E_i, I_i, D_i, Q_i, P_i$ and R_i , which definitions are gathered in Table 1.

Moreover, we define the human living population H_i and the total population N_i , of patch i , as

$$\begin{aligned} H_i &= S_i + E_i + I_i + Q_i + P_i + R_i, \\ N_i &= S_i + E_i + I_i + D_i + Q_i + P_i + R_i, \end{aligned} \quad (1)$$

respectively.

For a given patch i , susceptible (latent) individuals wish to travel to patch j at the rate a_{ji} . At the end of the patch i , because they are suspected of being infected with EVD due to previous contacts with suspected EVD cases [7] or because they undergo malaria-like symptoms, they are positively screened at the rate $\eta_i^S(\eta_i^E)$ and sent to quarantine. Thus, a number

$$\sum_{\substack{j=1 \\ j \neq i}}^n a_{ji} \eta_i^E E_i + \sum_{\substack{j=1 \\ j \neq i}}^n a_{ji} \eta_i^S S_i$$

of travelers coming from S_i and E_i will be stopped due to exit screening.

Similarly, among the proportion a_{ij} of susceptible and latent individuals wishing to travel from patch j to patch i , only the number of $a_{ij}(1 - \eta_j^S)S_j$ and

$a_{ij}(1 - \eta_j^E)E_j$, respectively, make the trip. At the entry of patch i , they undergo an optimal entry screening. Let us denote by $\tau_i(t)$, the time-dependent rate at which the entry screening is implemented in patch i . Based on the second assumption, none of the susceptibles are stopped, while a number $a_{ij}(1 - \eta_j^E)\tau_i(t)E_j$ of latent individuals are sent to quarantine, leading to the number

$$\sum_{\substack{j=1 \\ j \neq i}}^n a_{ij}(1 - \eta_j^E)\tau_i(t)E_j$$

of all latent individuals coming from the other patches, who are placed in quarantine in patch i .

As the other human living compartments, the quarantined individuals in patch i undergo natural mortality at the rate μ_i . Moreover, due to diagnosis tests in the Q_i compartment, individuals leave this compartment at the rate v_i . Hence the equation that governs the dynamics of Q_i is given as

$$\begin{aligned} \dot{Q}_i(t) &= \sum_{\substack{j=1 \\ j \neq i}}^n a_{ji} \eta_i^E E_i + \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij}(1 - \eta_j^E)\tau_i(t)E_j \\ &+ \sum_{\substack{j=1 \\ j \neq i}}^n a_{ji} \eta_i^S S_i - (\mu_i + v_i)Q_i. \end{aligned}$$

We define by $\phi_i(E, S_i)$ as the proportion of $v_i Q_i$ individuals who exit the quarantine compartment due to a positive laboratory test such that a number $\phi_i(E, S_i)v_i Q_i$ of quarantined individuals are sent in isolation. In isolation settings, apart from natural death, infected can recover at the rate θ_i or die due to the disease at the rate ψ_i . The dynamics of P_i is given as

$$\frac{dP_i(t)}{dt} = \phi_i(E, S_i)v_i Q_i - (\mu_i + \psi_i + \theta_i)P_i.$$

Being negatively diagnosed, the remaining number, $(1 - \phi_i(E, S_i))v_i Q_i$, of tested individuals are reverted to the susceptible compartments in patches. More precisely, among those who leave the Q_i compartment, $(1 - \phi_i(E, S_i))v_i \xi_{ii} Q_i$ cancel their trip (at the cancellation rate ξ_{ii}) due to delay and thus return to the compartment S_i , while $(1 - \phi_i(E, S_i))v_i \xi_{ji} Q_i$ leave the patch i to the patch j ($j \neq i$), with ξ_{ji} the rate to travel from the Q_i compartment to the S_j compartment. For each of such travelers from Q_i , the probability ξ_{ji} is such that $\sum_{j=1}^n \xi_{ji} = 1$.

Taking into account the natural death rate, the recruitment through births in the population denoted by

Parameters	Epidemiological interpretation	Units
a_{ij}	Rate of susceptible/latent individuals of patch j who wish to migrate to patch i .	week^{-1}
μ_i	Natural mortality rate in patch i .	week^{-1}
Λ_i	Constant recruitment of susceptible individuals in patch i .	indiv. week^{-1}
β_i	Effective transmission rate of EVD in patch i due to individuals in I_i compartment.	indiv. week^{-1}
b_i	Burial rate of Ebola-deceased in patch i .	week^{-1}
$\gamma_i(\theta_i)$	Recovery rate of infected who belong to the $I_i(P_i)$ compartment.	week^{-1}
α_i	Exit rate of the E_i compartment to the I_i compartment.	week^{-1}
ν_i	Modification parameter for the infectiousness of the Ebola-deceased.	
η_i^S	Proportion of susceptible individuals in S_i who are positively screened at the exit border of patch i .	
η_i^E	Proportion of latent individuals in E_i who are positively screened at the exit border of patch i .	
τ_i	Proportion of latent individuals of the other patches who are positively screened at the entry of patch i .	
δ_i	Mortality rate due to EVD of infected individuals in patch i who belong to the I_i compartment.	week^{-1}
ψ_i	Mortality rate due to EVD of infected in patch i who belong to the P_i compartment.	week^{-1}
v_i	Exit rate from the Q_i compartment by any means different from the death.	week^{-1}
$\phi_i(E_i)v_i$	Fraction of quarantined individuals who are positively diagnosed.	week^{-1}
ξ_{ij}	Rate at which the quarantined who are negatively diagnosed in patch j left patch j to patch i ($j = 1, \dots, n, j \neq i$).	week^{-1}
ξ_{ii}	Rate at which the quarantined who are negatively diagnosed in patch i cancel their trip.	week^{-1}

Table 2: Parameters of the model.

Λ_i , and the EVD infection force in the patch i defined as

$$\lambda_i \equiv \lambda_i(t) := (\beta_i(I_i + \nu_i D_i))/N_i,$$

the dynamics of S_i is given by:

$$\begin{aligned} \dot{S}_i(t) = & \Lambda_i - \lambda_i S_i - \mu_i S_i + \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij}(1 - \eta_j^S) S_j \\ & - \sum_{\substack{j=1 \\ j \neq i}}^n a_{ji} S_i + \sum_{j=1}^n (1 - \phi_j) v_j \xi_{ij} Q_j. \end{aligned}$$

Once infected, susceptible individuals move to the E_i compartment. Due to exit screening implemented in the other patches followed by entry screening in patch i , the number of latent migrants in patch i coming from the other patches is given as

$$\sum_{\substack{j=1 \\ j \neq i}}^n a_{ij}(1 - \eta_j^E)(1 - \tau_i(t)) E_j.$$

We denote by α_i the exit rate from E_i to I_i . Taking into account the number $\sum_{j=1, j \neq i}^n a_{ji} E_i$ of latent individuals in patch i who wish to travel, and the natural

mortality, the dynamics of E_i is governed by

$$\begin{aligned} \dot{E}_i = & \lambda_i S_i - \mu_i E_i - \alpha_i E_i \\ & + \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij}(1 - \eta_j^E)(1 - \tau_i(t)) E_j - \sum_{\substack{j=1 \\ j \neq i}}^n a_{ji} E_i. \end{aligned}$$

Symptomatic EVD infected are supplied by the $\alpha_i E_i$ individuals who left the E_i compartment. They can recover at rate γ_i , die naturally or due to the disease at the rate δ_i . The rate of change of I_i is given as

$$\dot{I}_i(t) = \alpha_i E_i - (\mu_i + \delta_i + \gamma_i) I_i.$$

Based on the fifth assumption, $(\mu_i + \delta_i) I_i$ individuals who left the compartment I_i are infectious and thus gathered in the D_i compartment. Individuals in D_i are buried at a rate b_i . The evolution of the Ebola infectious deceased is patch i is governed by the differential equation

$$\dot{D}_i(t) = (\mu_i + \delta_i) I_i - b_i D_i.$$

Recovered individuals coming from P_i and R_i are placed in the R_i compartment. In this compartment they undergo mortality as well, leading to the dynamics of R_i as

$$\dot{R}_i(t) = \gamma_i I_i + \theta_i P_i - \mu_i R_i.$$

We gather the definitions of the model parameters in Table 2. Putting all the equations together, $\forall i = 1, \dots, n$,

leads to the control System (2), for which the flow diagram for two patches is displayed in Figure 1.

The model:

$$\begin{aligned}
 \dot{S}_i(t) &= \Lambda_i - \beta_i \left(\frac{S_i I_i}{N_i} + \nu_i \frac{S_i D_i}{N_i} \right) - \mu_i S_i \\
 &\quad + \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij} (1 - \eta_j^S) S_j - \sum_{\substack{j=1 \\ j \neq i}}^n a_{ji} S_i \\
 &\quad + \sum_{j=1}^n (1 - \phi_j) v_j \xi_{ij} Q_j, \\
 \dot{E}_i(t) &= \beta_i \left(\frac{S_i I_i}{N_i} + \nu_i \frac{S_i D_i}{N_i} \right) - \mu_i E_i - \alpha_i E_i \\
 &\quad + \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij} (1 - \eta_j^E) (1 - \tau_i(t)) E_j - \sum_{\substack{j=1 \\ j \neq i}}^n a_{ji} E_i, \\
 \dot{I}_i(t) &= \alpha_i E_i - (\mu_i + \delta_i + \gamma_i) I_i, \\
 \dot{D}_i(t) &= (\mu_i + \delta_i) I_i - b_i D_i, \\
 \dot{Q}_i(t) &= \sum_{\substack{j=1 \\ j \neq i}}^n a_{ji} \eta_i^E E_j + \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij} (1 - \eta_j^E) \tau_i(t) E_j \\
 &\quad + \sum_{\substack{j=1 \\ j \neq i}}^n a_{ji} \eta_i^S S_j - (\mu_i + v_i) Q_i, \\
 \dot{P}_i(t) &= \phi_i v_i Q_i - (\mu_i + \psi_i + \theta_i) P_i, \\
 \dot{R}_i(t) &= \gamma_i I_i + \theta_i P_i - \mu_i R_i.
 \end{aligned} \tag{2}$$

We define the function $\phi_i \equiv \phi_i(E_1, \dots, E_n, S_i) : \mathbb{R}_+^n \times \mathbb{R}_+ \rightarrow [0, 1]$ as

$$\phi_i = \left(\sum_{p=1}^n E_p \right) / \left(S_i + \sum_{p=1}^n E_p \right).$$

The function ϕ_i reflects in some way the composition of the Q_i compartment and can be seen as the probability for a quarantined individual to be infected. This function reduces to $\phi_i = E_i / (S_i + E_i)$, in the case dealt with in [10] where the entry screening intervention is not implemented and thus, no latent travelers from the other patches are quarantined into patch i .

B. Theoretical analysis of Model (2)

Though the focus of this paper is on the optimal control problems associated with Model (2), we state in this section the results on the well-posedness and stability of Model (2).

For mathematical tractability, we assume in this section that the entry screening is constant. Moreover, apart from the definitions in (1), we add the notation

below that will be used throughout the paper. We denote the sums on all patches in bold uppercase letters, i.e.,

$$\begin{aligned}
 \mathbf{H} &= \sum_{i=1}^n H_i, \quad \mathbf{D} = \sum_{i=1}^n D_i, \quad \mathbf{\Lambda} = \sum_{i=1}^n \Lambda_i, \\
 \Upsilon &= \sum_{i=1}^n (\mu_i + \delta_i), \quad b_m = \min_{1 \leq i \leq n} \{b_i\}, \quad b^M = \max_{1 \leq i \leq n} (b_i), \\
 a^M &= \max_{1 \leq i \leq n} (\beta_i (1 + \nu_i) + \mu_i + \sum_{j=1}^n a_{ji}); \\
 \alpha^M &= \max_{1 \leq i \leq n} (\mu_i + \alpha_i), \quad \mu^M = \max_{1 \leq i \leq n} (\mu_i), \\
 v^M &= \max_{1 \leq i \leq n} (\mu_i + v_i), \quad \theta^M = \max_{1 \leq i \leq n} (\mu_i + \psi_i + \theta_i), \\
 k_i &= (\mu_i + \delta_i + \gamma_i); \quad \gamma^M = \max_{1 \leq i \leq n} (\mu_i + \delta_i + \gamma_i).
 \end{aligned}$$

Given a compartment in patch i , the initial letter / acronym of that compartment will denote the vector function having as components the associated compartments of the n patches. Hence

$$S = (S_1, S_2, \dots, S_n).$$

A calligraphic letter such as $\mathcal{X}$ denotes the $7n$ -vector-function given by

$$\mathcal{X} = (S, E, I, D, Q, P, R). \tag{3}$$

Furthermore, considered for all n patches, Model (2) can be written in the compact form,

$$\frac{d\mathcal{X}}{dt} = \mathcal{G}(\mathcal{X}),$$

where $\mathcal{X} \equiv \mathcal{X}(t)$, as in (3), denotes the solution of the system with right-hand side, $\mathcal{G} : \mathbb{R}^{7n} \setminus \{0\} \rightarrow \mathbb{R}^{7n}$, structured as

$$\begin{aligned}
 \mathcal{G}(\mathcal{X}) &= (G_{S_1}, \dots, G_{S_n}, G_{E_1}, \dots, G_{E_n}, \\
 &\quad G_{I_1}, \dots, G_{I_n}, G_{D_1}, \dots, G_{D_n}, G_{Q_1}, \dots, \\
 &\quad G_{Q_n}, G_{P_1}, \dots, G_{P_n}, G_{R_1}, \dots, G_{R_n})^T,
 \end{aligned} \tag{4}$$

where G_{A_i} represents the right-hand side of the equation of the dynamics of the compartment A_i , $A = S, E, I, D, Q, P, R$.

The announced result with proof similar to that in [10] reads as follows.

Theorem 1. Assume that the control, $\tau_i \equiv \tau_i(t)$, is a constant $\bar{\tau}_i$, $\forall i = 1, \dots, n$.

1) Model (2) is a dynamical system on the following biologically feasible and attractive region

$$\begin{aligned}
 \Gamma &:= \{ \mathcal{X} = (S, E, I, D, Q, P, R) \in \mathbb{R}_+^{7n} : \\
 &\quad \mathbf{H} \leq \mathbf{\Lambda} / \mu_m \text{ and } \mathbf{D} \leq (\Upsilon \mathbf{\Lambda}) / (\mu_m b_m) \}.
 \end{aligned}$$

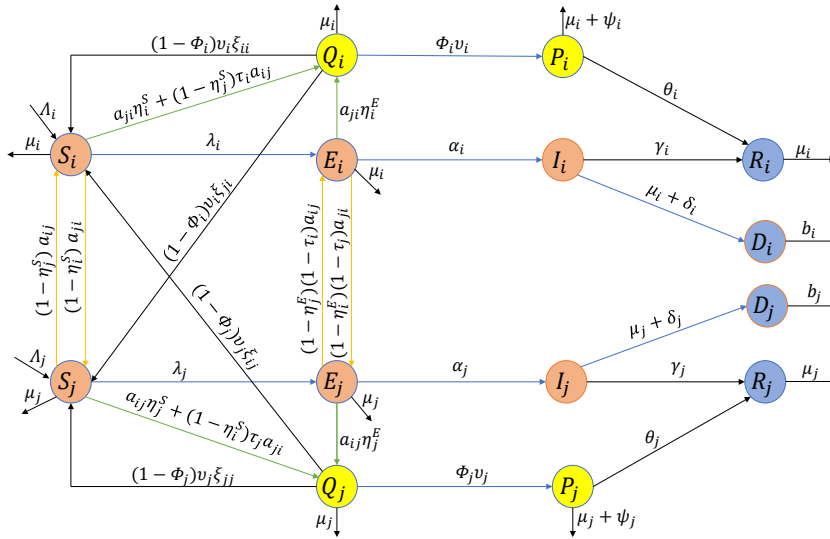


Fig. 1: Flow diagram between two infected patches i and j ($i \neq j$) for Model (2).

2) Model (2) satisfies the conservation laws

$$\begin{aligned} \frac{d\mathbf{H}}{dt} &= \sum_{i=1}^n (\Lambda_i - \mu_i H_i - \delta_i I_i - \psi_i P_i) \leq \mathbf{\Lambda} - \mu_m \mathbf{H}, \\ \frac{d\mathbf{D}}{dt} &= \sum_{i=1}^n ((\mu_i + \delta_i) I_i - b_i D_i) \leq \frac{\Upsilon \mathbf{\Lambda}}{\mu_m} - b_m \mathbf{D}. \end{aligned} \quad (5)$$

3) Model (2) has a unique disease-free equilibrium,

$$\mathcal{E}_0 := (S^0, 0, 0, 0, 0, Q^0, 0, 0) \in \mathbb{R}_+^{7n},$$

where (S_i^0, Q_i^0) , $i = 1, 2, \dots, n$, are suitably and uniquely determined.

4) The control reproduction number $\mathcal{R}_c$ is given by

$$\mathcal{R}_c = \rho(FV^{-1}) = \rho\left(-F_{12}V_{22}^{-1}V_{21}V_{11}^{-1} + F_{13}V_{33}^{-1}V_{32}V_{22}^{-1}V_{21}V_{11}^{-1}\right),$$

where

$$\begin{aligned} F_{12} &= \text{diag}\left(\beta_i \frac{S_i^0}{S_i^0 + Q_i^0}\right), \quad V_{32} = \text{diag}(-\mu_i - \delta_i), \\ F_{13} &= \text{diag}\left(\beta_i \nu_i \frac{S_i^0}{S_i^0 + Q_i^0}\right), \quad V_{21} = \text{diag}(-\alpha_i), \\ V_{22} &= \text{diag}(\mu_i + \delta_i + \gamma_i), \quad V_{33} = \text{diag}(b_i), \\ V_{11} &= \text{diag}\left(\sum_{j \neq i}^n a_{ji} + \mu_i + \alpha_i\right) \\ &\quad - (a_{ij}(1 - \eta_j^E)(1 - \bar{\tau}_i)), \end{aligned}$$

with $1 \leq i, j \leq n$.

5) The DFE, $\mathcal{E}_0$, is locally asymptotically stable (LAS) whenever $\mathcal{R}_c < 1$, and it is unstable if $\mathcal{R}_c > 1$.

6) The DFE is globally asymptotically stable (GAS) in Γ whenever $\mathcal{T} < 1$, where

$$\mathcal{T} = \max_{1 \leq i \leq n} (\mathcal{T}^i) \quad (6)$$

with

$$\begin{aligned} \mathcal{T}^i &= (\beta_i \alpha_i (b_i + \nu_i (\mu_i + \delta_i)) \\ &\quad + b_i k_i \sum_{\substack{j=1 \\ j \neq i}}^n a_{ji} \eta_i^E \bar{\tau}_j) / (b_i k_i (\mu_i + \alpha_i)). \end{aligned}$$

Equation (6) demonstrates that the threshold $\mathcal{T}$ increases with migration rates, highlighting the importance of implementing border measures to effectively control EVD.

C. Optimal entry screening

We choose the following cost/objective functional:

$$\begin{aligned} J &\equiv J(x, \tau) = \int_0^T g(t, x, \tau_i) dt, \\ g(t, x, \tau_i) &:= \sum_{i=1}^n \left(w_i E_i(t) + \frac{c_i}{2} \tau_i^2(t) \right), \end{aligned} \quad (7)$$

where T is the maximum duration of implementation of the entry screening measure in the n patches, w_i is the weight factor associated with latent humans E_i , and c_i the weight factor linked to the control variable $\tau_i(t)$ (assumed to be Lebesgue measurable). The term

$w_i E_i$ represents the cost of infection of latent cases E_i , whereas the term $c_i \tau_i^2/2$ is the cost of entry screening in patch i needed to minimize the number of EVD-infected. For the reader's convenience, we introduce the notation of the model variables in the following compact form:

$$\begin{aligned} x_i &\equiv \mathcal{X}_i = (x_{1i}, x_{2i}, x_{3i}, x_{4i}, x_{5i}, x_{6i}, x_{7i}) \\ &= (S_i, E_i, I_i, Q_i, P_i, D_i, R_i), \quad \forall i = 1, \dots, n, \\ x &\equiv \mathcal{X} = (x_{11}, x_{12}, \dots, x_{1n}, x_{21}, x_{22}, \dots, x_{2n}, \dots, \\ &\quad x_{71}, x_{72}, \dots, x_{7n}). \end{aligned} \quad (8)$$

We have the following result on the existence and uniqueness of the optimal problem.

Theorem 2. *There exists a unique control*

$$\tau^* := (\tau_1^*, \dots, \tau_n^*) \in U := [0, 1]^n := [0, 1] \times \dots \times [0, 1]$$

(n times) and its associated trajectory x_τ^* such that the objective functional in (7) is minimized, i.e.,

$$J(x_\tau^*, \tau^*) = \min_{\tau \in U} J(x, \tau). \quad (9)$$

Proof: We establish the existence of the optimal control vector $\tau^* = (\tau_1^*, \dots, \tau_n^*)$ and its associated optimal trajectory x^* for the optimization problem (2) which consists in the minimization of the objective functional (7) over the set $U = [0, 1]^n := [0, 1] \times \dots \times [0, 1]$ (n times). Let us recall that if $\tau \in U$, then the pair (x_0, τ) is called feasible if there exists a solution for the control System (2) in the interval $[0, T]$, with the initial condition $x(0) = x_0$ and the final condition $x(T)$ [27].

Clearly, $\mathcal{F}$, the right-hand side of (2), is a $\mathcal{C}^1$ function. Moreover, since both the control set $U = [0, 1]^n$ and the cost functional are convex, to prove the existence of each optimal control τ_i^* , it is sufficient to verify that there exist constants $C, C_{1i}, C_{2i}, C_{3i}, C_{4i} > 0, r_i \geq 0, i = 1, \dots, n$ such that the following conditions hold [26, 27]:

1. (a) $\|\mathcal{F}_{\tau_i}(t, x, \tau)\| \leq C_{1i}, \quad \forall i$.
 (b) $\|\mathcal{F}(t, 0, 0)\| \leq C$.
 (c) $\|\mathcal{F}_x(t, x, \tau)\| \leq C_{4i}(1 + \|\tau_i\|),$
 $C_{4i} > 0, \quad \forall i = 1, \dots, n$.
2. The set $\mathcal{M}$ of feasible controls is non empty.
3. $\mathcal{F}(t, x, \tau) = h(t, x) + v_i(t, x)\tau_i, \quad \forall i = 1, \dots, n$.
4. $g(t, x, \tau) \geq C_{2i}|\tau_i|^{r_i} - C_{3i}$ with $C_{2i} > 0$ and $r_i > 1$.

Indeed, we obviously have

$$\begin{aligned} \mathcal{F}_{\tau_i}(t, x, \tau) &= (\underbrace{0, \dots, 0}_{n \text{ times}}, \underbrace{0, \dots, 0}_{i-1 \text{ times}}, \\ &\quad - \sum_{j=1}^n a_{ij}(1 - \eta_j^E)E_j, \underbrace{0, \dots, 0}_{n-i \text{ times}}, \underbrace{0, \dots, 0}_{2n \text{ times}}, \underbrace{0, \dots, 0}_{i-1 \text{ times}}, \\ &\quad \sum_{j=1}^n a_{ij}(1 - \eta_j^E)E_j, \underbrace{0, \dots, 0}_{n-i \text{ times}}, \underbrace{0, \dots, 0}_{2n \text{ times}}, 0)^T, \end{aligned}$$

so that the Euclidean norm of $\mathcal{F}_\tau$ satisfies the following inequality:

$$\begin{aligned} \|\mathcal{F}_{\tau_i}(t, x, \tau)\| &= \sqrt{2 \left(\sum_{j=1}^n a_{nj}(1 - \eta_j^E)E_j \right)^2} \\ &\leq \sqrt{2} \omega_0 \sum_{j=1}^n a_{nj}(1 - \eta_j^E), \end{aligned}$$

with $\omega_0 := \Lambda/\mu_m$.

Thus, taking

$$C_{1i} = \sqrt{2} \omega_0 \sum_{j=1}^n a_{ij}(1 - \eta_j^E), \quad \forall i = 1, \dots, n,$$

proves assertion 1.(a). The assertions 1.(b) and 1.(c) can be proven similarly. On the other hand, since f is $\mathcal{C}^1$, then the Cauchy-Lipschitz theorem guarantees the existence of a unique solution of the initial value problem $\dot{x} = f(t, x, 0), x(0) = x_0$. Hence, $\mathcal{M}$ is not empty. This proves the condition 2. As for condition 3, take

$$\begin{aligned} v_i(t, x) &= \mathcal{F}_{\tau_i}(t, x, \tau) \\ h(t, x) &= (f_{S_1}, \dots, f_{S_n}, h_{E_1}, \dots, h_{E_n}, f_{I_1}, \dots, f_{I_n}, \\ &\quad f_{D_1}, \dots, f_{D_n}, h_{Q_1}, \dots, h_{Q_n}, f_{P_1}, \dots, f_{P_n}, \\ &\quad f_{R_1}, \dots, f_{R_n})^T, \\ h_{E_i} &= \beta_i(S_i I_i + \nu_i S_i D_i)/(N_i) - \mu_i E_i - \alpha_i E_i \\ &\quad + \sum_{j=1}^n a_{ij}(1 - \eta_j^E)E_j - \sum_{j=1}^n a_{ji}E_i \\ h_{Q_i} &= \sum_{j=1}^n a_{ji}\eta_i^E E_j + \sum_{j=1}^n a_{ji}\eta_i^S S_j - (\mu_i + \nu_i)Q_i, \\ &\quad \forall i = 1, \dots, n. \end{aligned}$$

Furthermore, one has

$$g(t, x, \tau) \geq \frac{c_i}{2} \tau_i^2(t), \quad \forall i = 1, \dots, n.$$

Thus, taking $r_i = 2, C_{2i} = c_i/2$ and $C_{3i} = 0, \forall i = 1, \dots, n$, yields condition 4. This achieves the proof of

the existence of the optimal entry screening control τ_i^* , $\forall i = 1, \dots, n$ and its associated optimal trajectory. ■

The characterization of the optimal control comes from the Pontryagin maximum/minimum principle. The Hamiltonian $\mathcal{H}$ of the System (2)–(7) is defined as

$$\mathcal{H}(t, x, \tau, z) = g(t, x, \tau) + \langle z | \mathcal{F}(t, x, \tau) \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the inner product of $\mathbb{R}^n$ and the function $\mathcal{F}(t, x, \tau)$, defined on $\mathbb{R}^{7n} \setminus \{0\}$ as in (4), is the right-hand side of Eq. (2)

$$\begin{aligned} \mathcal{F}(t, x, \tau) = & (f_{S_1}, \dots, f_{S_n}, f_{E_1}, \dots, f_{E_n}, \\ & f_{I_1}, \dots, f_{I_n}, f_{D_1}, \dots, f_{D_n}, f_{Q_1}, \dots, f_{Q_n}, \\ & f_{P_1}, \dots, f_{P_n}, f_{R_1}, \dots, f_{R_n})^T, \end{aligned}$$

and $z \in \mathbb{R}^{7n}$ is a vector whose coordinates are written as in the second equation in (8). The Hamiltonian (20) and the detailed system (21) of ordinary differential equations for the adjoint variables are in the Appendix.

One has the following result, proved for instance in [28]:

Theorem 3. Assume $\mathcal{F}(t, x, \tau)$ and $g(t, x, \tau)$ are continuously differentiable in t, x and τ , and convex in x and τ . Let τ^* be an optimal control to the control problem defined by (2) with an associated state x_τ^* . Then there exists a differentiable adjoint function $z(t) : [0, T] \rightarrow \mathbb{R}^{7n}$ such that the following three properties hold:

- 1) The optimality condition: $\mathcal{H}(t, x, \tau) \geq \mathcal{H}(t, x_\tau^*, \tau^*)$, for all controls τ at any time $t \in [0, T]$.
- 2) The adjoint equation: $\dot{z}_{li} = -\frac{\partial \mathcal{H}}{\partial x_{li}}, i = 1, \dots, n$.
- 3) The transversality condition:

$$z_{li}(T) = 0, l = 1, \dots, 7, i = 1, \dots, n.$$

Remark 1. Considering the differentiability of the Hamiltonian $\mathcal{H}$ and the fact that, our optimal control is of minimization type, $\mathcal{H}$ verifies the two conditions

$$\frac{\partial \mathcal{H}}{\partial \tau_i} = 0, \quad \frac{\partial^2 \mathcal{H}}{\partial \tau_i^2} > 0, \text{ at } \tau_i^*, \forall t > 0, i = 1, \dots, n.$$

In view of the bounds of the control τ_i , we take for $i \in \{1, \dots, n\}$:

$$\begin{cases} \tau_i^* = 1, & \text{if } \frac{\partial \mathcal{H}}{\partial \tau_i} < 0, \\ \tau_i^* \in [0, 1], & \text{if } \frac{\partial \mathcal{H}}{\partial \tau_i} = 0, \\ \tau_i^* = 0, & \text{if } \frac{\partial \mathcal{H}}{\partial \tau_i} > 0. \end{cases}$$

Differentiating $\mathcal{H}$ with respect to τ_i gives

$$\frac{\partial \mathcal{H}_i}{\partial \tau_i} = c_i \tau_i + \sum_{j=1, j \neq i}^n a_{ij} (1 - \eta_j^E) E_j (z_{5i} - z_{2i}).$$

Let $T_{\tau_i}^*$ be the largest time in the interval $[0, T]$, which satisfies the condition $\frac{\partial \mathcal{H}}{\partial \tau_i} < 0$ and $T_{\tau_i}^{**}$ be the smallest time in the interval $[T_{\tau_i}^*, T]$, that satisfies the relation $\frac{\partial \mathcal{H}}{\partial \tau_i} > 0$. Then the optimal control τ_i^* , $\forall i = 1, \dots, n$, can be written as

$$\tau_i^*(t) = \begin{cases} 1, & \text{if } 0 \leq t < T_{\tau_i}^* \\ \frac{(z_{2i} - z_{5i})}{c_i} \sum_{j=1, j \neq i}^n a_{ij} (1 - \eta_j^E) E_j, & \text{if } T_{\tau_i}^* \leq t \leq T_{\tau_i}^{**}, \\ 0, & \text{if } T_{\tau_i}^{**} < t \leq T. \end{cases} \quad (10)$$

We now come to the numerical discretization of the optimal control problem (2) and (9). Considering the rich dynamics of the state variable model summarized in Theorem 1, it is essential to use a reliable numerical method that preserves these qualitative properties. Typically, classical methods such as linear multi-step and explicit Runge-Kutta methods fail to do so as widely studied in the literature (see, for instance, [29–31]). Consequently, we design a Nonstandard Finite Difference (NSFD) scheme, an approach created by R. Mickens in the late eighties and that has shown great potential in producing numerical schemes that are dynamically consistent with the continuous model (see [32, 33] for ODEs, and [34, 35] for optimal control problems).

Let p be a positive integer. We divide the interval $[0, T]$ into a finite number of sub-intervals with step size $\Delta t =: T/p$ through the nodes $t_k := k\Delta t$, $k = 0, \dots, p$. For $l = 1, \dots, 7$, $i = 1, \dots, n$, we denote by x_{li}^k , z_{li}^k and τ_i^k the approximations of $x_{li}(t_k)$, $z_{li}(t_k)$ and $\tau_i(t_k)$, respectively. For the iterations $m \geq 0$, we choose an initial guess $\tau^0 = \tau^0(t)$. The purpose of the continuous Forward Backward Sweep Method (FBSM) is to find the iterations x^{m+1} , z^{m+1} and τ^{m+1} as solutions to the system:

$$\begin{aligned} \frac{dx^{m+1}(t)}{dt} &= \mathcal{F}(t, x^{m+1}(t), \tau^m), \\ \frac{dz^{m+1}(t)}{dt} &= \Psi(t, x^{m+1}(t), z^{m+1}(t), \tau^m(t)), \\ \tau^{m+1}(t) &= \varphi(t, x^{m+1}(t), z^{m+1}(t)), \\ x^{m+1}(0) &= x(0) \equiv x_0, \quad z^{m+1}(T) = 0, \end{aligned} \quad (11)$$

where

$$\begin{aligned} \Psi(t, x^{m+1}(t), z^{m+1}(t), \tau^m(t)) &= -g_x(t, x^{m+1}(t), \\ &\quad \tau^m(t)) - \langle z^{m+1}(t), \mathcal{F}_x(t, x^{m+1}(t), \tau^m(t)) \rangle, \end{aligned}$$

and the function φ , that defines $\tau^{m+1}(t)$, comes from (10).

We define the algorithm to approximate the systems in (11) as follows:

- Step 1. For $m = 0$, we choose an initial guess $\tau^{k,0}$, $k = 0, 1, \dots, p$.
- Step 2. For $m \geq 0$, we compute $x^{k+1,m+1}$ by the NSFD forward Euler scheme:

$$\frac{x^{k+1,m+1} - x^{k,m+1}}{r} = \mathcal{F}(t_k, x^{k,m+1}, \tau^{k,m}), \quad (12)$$

$$x_0^{k,m+1} = x_0^k, \quad k = 0, 1, \dots, p-1.$$

Here, the nontrivial denominator function r is defined by

$$r \equiv r(h) = \frac{1 - e^{-qh}}{q} = h + \mathcal{O}(h^2),$$

such that

$$q \geq \max\{v^M, \sum_{i,j=1}^n a_{ji}, a^M, \alpha^M, \gamma^M, b^M, \theta^M, \mu^M\}$$

$$\text{and } q > \frac{|\vartheta|^2}{2|\Re \vartheta|},$$

where ϑ is any eigenvalue of the Jacobian matrix of Model (2) at the DFE.

Furthermore, we use the new state values to define the discrete adjoint solutions, $z^{k+1,m+1}$, denoted as in (8), by the classical backward Euler scheme: for $l = 1, \dots, 7$, $i = 1, \dots, n$

$$\frac{z^{k+1,m+1} - z^{k,m+1}}{\Delta t} = \Psi(t_{k+1}, x^{k+1,m+1}, z^{k+1,m+1}, \tau^{k+1,m}), \quad (13)$$

$$z^{p,m+1} = 0, \quad k = p-1, \dots, 0.$$

Note that the forward-backward-sweep strategy is used, because (11) is a two-point boundary value problem, the numerical discretizations of which are typically done by special techniques such as the shooting method.

- Step 3. We update the discrete control by

$$\tau^{k+1,m+1} = \varphi(t_{k+1}, x^{k+1,m+1}, z^{k+1,m+1}).$$

- Step 4. We stop the process at the iteration m once the values of the unknowns at the previous iteration are very close to the ones at the present iteration, as usual, see for instance [28]. Otherwise, we repeat the process in the loop from Step 2 with the iteration $m+1$.

The following result that can be easily derived is illustrated on Figure 2:

Theorem 4. Assume that the control, $\tau \equiv \tau(t)$, is a constant $\bar{\tau}$. Then, the NSFD scheme (12) or (22)

preserves the properties gathered in Theorem 1 in the specific form of the following items:

- 1) The non-negativity and upper bounds of the continuous state variables model (2)
- 2) The local asymptotic stability of the DFE if $\mathcal{R}_c < 1$, and its instability if $\mathcal{R}_c > 1$.
- 3) The global asymptotic stability of the DFE of the continuous model if $\mathcal{T} < 1$.

Unlike the state variables system (2) or (11) from which the NSFD schemes inherited the qualitative features stated in Theorem 1, no particular properties need to be preserved by the associated adjoint system. Hence, its discretization by the classical backward Euler scheme in Step 2 above.

The detailed expression of the NSFD scheme (12) for Model (11) is given in Eq. (22) for $i = 1, \dots, n$. Finally, the classical scheme (13) for the adjoint system reads as in Eq. (23).

III. CONTROL THROUGH OPTIMAL EXIT SCREENING AND VACCINATION

In this section, we drop the entry screening from Model (2) and propose a model with an optimal exit screening. But, because the implementation of the exit screening in a given patch i seems to be more beneficial to other patches, we combine this measure with the vaccination (there are currently two licensed vaccines, namely the Zabdeno (Ad26.ZEBOV) and Mvabea (MVA-BN-Filo) [36]). Thus, we assume that the exit screening implemented in patch i to target the latent travelers is a time-dependent function $\eta_i(t)$, while $\sigma_i(t)$ is the time-dependent rate for the vaccination in patch i . Since the EVD vaccine is imperfect, we assume that, among the $\sigma_i(t)S_i$ vaccinated individuals, only a fraction, κ , acquired immunity (i.e. reached the compartment R_i). With these two control measures, Model (2) is modified as follows for $i = 1, \dots, n$:

$$\dot{S}_i(t) = \Lambda_i - \beta_i(S_i I_i + \nu_i S_i D_i)/(N_i) - \mu_i S_i$$

$$+ \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij} S_j - \sum_{\substack{j=1 \\ j \neq i}}^n a_{ji} S_i - \kappa \sigma_i(t) S_i,$$

$$\dot{E}_i(t) = \beta_i(S_i I_i + \nu_i S_i D_i)/(N_i) - \mu_i E_i - \alpha_i E_i$$

$$+ \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij} (1 - \eta_j(t)) E_j - \sum_{\substack{j=1 \\ j \neq i}}^n a_{ji} E_i,$$

$$\begin{aligned}\dot{I}_i(t) &= \alpha_i E_i - (\mu_i + \delta_i + \gamma_i) I_i, \\ \dot{D}_i(t) &= (\mu_i + \delta_i) I_i - b_i D_i, \\ \dot{Q}_i(t) &= \sum_{\substack{j=1 \\ j \neq i}}^n a_{ji} \eta_j(t) E_i - (\mu_i + v_i) Q_i, \\ \dot{P}_i(t) &= v_i Q_i - (\mu_i + \psi_i + \theta_i) P_i, \\ \dot{R}_i(t) &= \kappa \sigma_i(t) S_i + \gamma_i I_i + \theta_i P_i - \mu_i R_i.\end{aligned}\quad (14)$$

We define the objective functional by

$$\begin{aligned}L &\equiv L(x, \sigma, \eta) \\ &= \int_0^T \left(\sum_{i=1}^n w_i E_i(t) + \frac{d_i}{2} \sigma_i^2(t) + \frac{c_i}{2} \eta_i^2(t) \right) dt,\end{aligned}\quad (15)$$

where, despite the presence of two vectors σ and η , the parameters T, w_i, d_i and c_i have similar meanings to the control problem in Section II. Hence, the optimal problem consists in finding the optimal trajectory $x_{\sigma, \eta}^*$ associated with the optimal control (σ^*, η^*) satisfying the controlled System (14) that minimizes the objective functional (15) on the set:

$$\begin{aligned}K &= \{(\sigma(t), \eta(t)) \text{ Lebesgue measurable}, \\ &0 \leq \sigma_i(t), \eta_i(t) \leq 1, \forall t \in [0, T], i = 1, \dots, n\}.\end{aligned}\quad (16)$$

Given the similarity in the reasoning and techniques with the entry screening control problem investigated in Section II, we shorten the presentation of the current section by merely stating the process and the results in two steps.

Firstly, for the characterization of the optimal control

$$(\sigma^*, \eta^*) = (\sigma_1^*, \dots, \sigma_n^*, \eta_1^*, \dots, \eta_n^*),$$

we need the Hamiltonian $\mathcal{H}$ of System (14)–(15) defined by

$$\mathcal{H}(t, x, z, \eta, \sigma) = g(t, x, \eta, \sigma) + \langle z | \mathcal{F}(t, x, \eta, \sigma) \rangle, \quad (17)$$

where $\mathcal{F}$ is defined by the right-hand side of System (14) and

$$g(t, x, \eta, \sigma) = \sum_{i=1}^n w_i E_i(t) + \frac{d_i}{2} \sigma_i^2(t) + \frac{c_i}{2} \eta_i^2(t).$$

We have the following result:

Theorem 5. *Let us assume that $\mathcal{F}(t, x, \eta, \sigma)$ and $g(t, x, \eta, \sigma)$ are continuously differentiable in t, x, η and σ , and convex in x, η and σ . If (η^*, σ^*) associated with the state variable, $x_{\eta, \sigma}^*$, is a solution of the optimal control problem defined in (14) and the control set in (16), there exists a differentiable adjoint function*

$z(t) : [0, T] \rightarrow \mathbb{R}^{7n}$, which verifies the following properties:

1) The optimality condition:

$$\mathcal{H}(t, x, \eta, \sigma) \geq \mathcal{H}(t, x_{\eta^*, \sigma^*}^*, \eta^*, \sigma^*),$$

for all controls η, σ at any time $t \in [0, T]$.

2) The adjoint equation: $\dot{z}_{li} = -\frac{\partial \mathcal{H}}{\partial x_{li}}, i = 1, \dots, n$.

3) The transversality condition:

$$z_{li}(T) = 0, \quad l = 1, \dots, 7, \quad i = 1, \dots, n.$$

The explicit expressions of the Hamiltonian $\mathcal{H}$ and the adjoint equations are provided in the Appendix.

Furthermore, we have the existence and uniqueness result below. The characterizations of σ^* and η^* are provided in the Appendix.

Theorem 6. *There exists a unique control $(\sigma^*, \eta^*) \in K$ such that the cost functional defined by (15) is minimized: $L(x_{\sigma, \eta}^*, \sigma^*, \eta^*) = \min_{(\sigma, \eta) \in K} L(x, \sigma, \eta)$. Moreover, defining the times $T_{\eta_i}^*, T_{\sigma_i}^*, T_{\eta_i}^{**}, T_{\sigma_i}^{**}$ as $T_{\tau_i}^*$ and $T_{\tau_i}^{**}$ as in Remark 1, these controls can be expressed as follows, $\forall i = 1, \dots, n$,*

$$\begin{aligned}\eta_i^*(t) &= \begin{cases} 1, & \text{if } 0 \leq t < T_{\eta_i}^* \\ \frac{1}{c_i} \sum_{\substack{j=1 \\ j \neq i}}^n a_{ji} (z_{2j} - z_{5i}) E_i, & \text{if } T_{\eta_i}^* \leq t \leq T_{\eta_i}^{**} \\ 0, & \text{if } T_{\eta_i}^{**} < t \leq T \end{cases} \\ \sigma_i^*(t) &= \begin{cases} 1, & \text{if } 0 \leq t < T_{\sigma_i}^* \\ \frac{S_i \kappa (z_{1i} - z_{7i})}{d_i}, & \text{if } T_{\sigma_i}^* \leq t \leq T_{\sigma_i}^{**} \\ 0, & \text{if } T_{\sigma_i}^{**} < t \leq T \end{cases}\end{aligned}\quad (18)$$

Secondly, we introduce the Nonstandard Forward-Backward Sweep Method for the exit screening and vaccination optimal control model by starting from the following iterative continuous system in $x^{m+1}, z^{m+1}, \eta^{m+1}, \sigma^{m+1}$:

$$\begin{aligned}\frac{dx^{m+1}(t)}{dt} &= \mathcal{F}(t, x^{m+1}(t), \eta^m(t), \sigma^m(t)), \\ \frac{dz^{m+1}(t)}{dt} &= \Psi(t, x^{m+1}(t), \eta^m(t), \sigma^m(t)), \\ \eta^{m+1}(t) &:= \varphi_1(t, x^{m+1}(t), z^{m+1}(t)), \\ \sigma^{m+1}(t) &:= \varphi_2(t, x^{m+1}(t), z^{m+1}(t)), \\ x^{m+1}(0) &= x(0), \quad z^{m+1}(T) = 0,\end{aligned}\quad (19)$$

where the functions φ_1 and φ_2 coming from (18) are

defined on $\mathbb{R}_+ \times \mathbb{R}_+^{7n} \times \mathbb{R}^{7n}$, and

$$\Psi(t, x^{m+1}(t), \eta^m(t), \sigma^m(t)) = -\left(g_x(t, x^{m+1}(t), \eta^m(t), \sigma^m(t)) - z^{m+1} \mathcal{F}_x(t, x^{m+1}(t), \eta^m(t), \sigma^m(t))\right).$$

Now, using the discretization of the interval $[0, T]$ introduced in Section II, the algorithm describing the fully discrete NSFD-FBSM works as follows:

- Step 1. For $m = 0$, we choose the initial guess $\eta^{k,0}$, $\sigma^{k,0}$, $k = 0, 1, \dots, p$.
- Step 2. For $m \geq 0$,:

- We compute $x^{k+1,m+1}$ by the NSFD forward Euler scheme as

$$\frac{x^{k+1,m+1} - x^{k,m+1}}{r} = \mathcal{F}(t_k, x^{k,m+1}, \eta^{k,m}, \sigma^{k,m}), \quad x_0^{k,m+1} = x_0^k, \quad k = 0, 1, \dots, p-1,$$

where r is defined in Section II.

- We compute $z^{k+1,m+1}$ by the classical backward Euler scheme: for $l = 1, \dots, 7$, $i = 1, \dots, n$ and $k = n-1, \dots, 0$ as:

$$\frac{z^{k+1,m+1} - z^{k,m+1}}{\Delta t} = \Psi(t_{k+1}, x^{k+1,m+1}, z^{k+1,m+1}, \eta^{k+1,m}, \sigma^{k+1,m}), \quad z^{p,m+1} = 0.$$

- Step 3. We update the discrete control by

$$\eta^{k+1,m+1} = \varphi_1(t_{k+1}, x^{k+1,m+1}, z^{k+1,m+1}), \\ \sigma^{k+1,m+1} = \varphi_2(t_{k+1}, x^{k+1,m+1}, z^{k+1,m+1}).$$

The iteration process stops as in Step 4. of the NSFD-FBSM, (12) and (13), or (22) and (23), presented in the previous section.

The detailed expression of the state variables of the NSFD-FBSM, in Step 2, for the System (14) is given by equation (24) for $i = 1, \dots, n$. Details for the discrete adjoint variables can be obtained from the right-hand side of the continuous model in Eq. (26).

IV. NUMERICAL SIMULATIONS

The numerical simulations are restricted to the three countries: Guinea (patch 1), Liberia (patch 2), and Sierra Leone (patch 3) where the 2014-2016 EVD outbreak was more pronounced (so that $n = 3$). Most of the parameters needed for the numerical simulations are given in Table 4, while the initial conditions are in Table 3.

Before giving numerical simulations for optimal control problems, we numerically illustrate the qualitative results stated in Theorem 4 as well as the analog theorem for the NSFD scheme for the continuous

	Guinea	Liberia	Sierra Leone
S(0)	11,055,429	4,248,000	6,964,859
	[37]	[38]	[39]
E(0)	330	1,319	1,262
I(0)	286	1,060	720
D(0)	286	1,060	720
Q(0)	286	1,060	720
P(0)	286	1,060	720
R(0)	286	1,060	720
Total	1,760	6,619	4,862

Table 3: Initial values of the variables for Model (2).

state variables model (14). We first show on Figure 2 that when τ, η and σ are fixed, the NSFD schemes for the control Systems (2) and (14) preserve the conservation law (5), contrarily to MATLAB *ode45* solver. Indeed, it is observed that *ode45* produces a total population of humans that is beyond its carrying capacity, while the NSFD schemes faithfully replicate both the non-negativity of the total population and its boundedness by the carrying capacity. Next, we have the self-explanatory top rows of Figures 3 and 4, which show the importance of having, for optimal control problems, numerical schemes that preserve the qualitative properties of the associated continuous variable systems. In particular, one observes on the said figures the preservation by the NSFD schemes of the global asymptotic stability of the disease-free equilibrium of the control Systems (2) and (14) when $\mathcal{T} < 1$ and $\mathcal{T}_{\bar{\eta}, \bar{\sigma}} < 1$, respectively.

In the bottom rows of Figures 3 and 4, we see that when $\mathcal{R}_c < 1 < \mathcal{T}$ and $\mathcal{R}_c < 1 = \mathcal{T}_{\bar{\eta}, \bar{\sigma}}$, respectively, the DFE might be globally asymptotically stable. This finding was not obtained theoretically. Furthermore, although it is challenging to compare $\mathcal{R}_c$ with $\mathcal{T}$ (or $\mathcal{R}_c$ with $\mathcal{T}_{\bar{\eta}, \bar{\sigma}}$), numerical computations using different sets of parameter values suggest that $\mathcal{R}_c < \mathcal{T}$ and $\mathcal{R}_c \approx \mathcal{T}_{\bar{\eta}, \bar{\sigma}}$ (see for instance the values displayed in the captions of Figures 3 and 4).

Let us now numerically solve the optimal control problems defined earlier in the previous sections. We do this in the following manner: We solve the optimality systems during forty weeks, using the Euler-based NSFD-FBSM described above. For comparison purposes, we also use the Euler-based classical FBSM.

In the top row of Figure 5, we provide the solution curves of the first optimal control problem as well as the curve without optimal entry screening with the classical Euler-based FBSM (using $h = 0.05$). In the bottom row, we plot the same curves by the Euler-based NSFD-FBSM (with $h = 1$ for the control model and $h = 0.05$

Par.	Value	Source	Par.	Value	Source
η_1^E	0.0239	[40]	η_2^E	0.0478	
μ_1	0.0002		v_1	0.2758	
β_1	0.2556		β_2	0.1209	
β_3	0.2822		v_2	0.2565	
ν_1	0.9374	[41]	ν_2	0.8524	[42]
ν_3	0.4044		η_3^S	0.1935	
δ_1	0.857		δ_2	0.75	
δ_3	0.5		v_3	0.3173	
ψ_1	0.3	[44]	ψ_2	0.4	[45]
ψ_3	0.5	[43]	$\xi_{ij}, \forall i, j$	1/3	Assumed
η_3^E	0.0578		a_{21}	0.000064	
a_{31}	0.0001		a_{12}	0.00054	
a_{32}	0.0001		a_{13}	0.00063	
a_{23}	0.000036		b_3	0.5	
b_1	1/2.01	[43]	b_2	1/4.5	[42]
γ_1	0.0059	[46]	γ_2	0.026767	[46]
θ_1	0.001120	[46]	θ_2	0.031486	[46]
θ_3	0.015743	[46]	γ_3	0.010038	[46]
η_2^S	0.0299	[40]	μ_2	0.0002	[40]
μ_3	0.0002		$\Lambda_i, \forall i$	0.03703	[47]
α_1	0.4127		α_2	0.4532	
α_3	1.9440		η_1^S	0.8535	

Table 4: Parameter values for simulations. Most of the parameters without source were estimated in [10].

for the adjoint system). The choice of different time step values for the two methods is motivated by the fact that classical explicit methods such as the forward Euler scheme in our case require smaller step sizes for stability [48], while NSFD schemes are unconditionally stable [29, 31]. More-so, whether the system is stiff or not, the nonstandard theta scheme is absolutely elementary stable as studied in [49]. The Euler-based NSFD-FBSM produces the same output irrespective of the time step used (so, the same result using the Euler-based NSFD-FBSM will be obtained for $h = 0.05$ for both the control and the adjoint systems).

Figure 5 should be read in conjunction with Table 5, where the times $T_{\tau_i}^*$ and $T_{\tau_i}^{**}$ needed to include the bounds of the control are indicated as explained in Eq. (10). Although weak, the bottom figures point out the usefulness of the optimal entry screening to reduce the burden of the disease. The control profiles τ_1^*, τ_2^* and τ_3^* show that the entry screening in Guinea, Liberia, and Sierra Leone should be at their highest values during 32.5, 18 and 2 weeks, respectively, before decreasing slowly to the lowest bound at the final time. On the contrary, the top figures display negative latent individuals, which is not acceptable. It also shows that all the control profiles reduce to the constant function zero, as if no control was implemented, which is not the case.

The second optimal control problem consists of three

subproblems:

- Optimal exit screening alone;
- Optimal vaccination alone;
- Optimal exit screening combined with optimal vaccination.

These scenarios are investigated in Figures 6, 7, and 8, respectively, using both the classical Euler-FBSM and the Euler-based NSFD-FBSM (with $h=2$). The top rows of these figures, obtained using the classical Euler-FBSM with ($h=0.05$), exhibit the same shortcomings previously observed in Figure 5. In contrast, the bottom rows, corresponding to the Euler-based NSFD-FBSM, clearly highlight the effectiveness of the associated control measures in reducing the spread of EVD.

In particular, Figure 7 shows that vaccination alone is a highly effective intervention for mitigating EVD transmission, thereby supporting its extensive implementation during the 2018–2020 DRC Kivu outbreak [25]. By comparison, Figure 6 reveals that exit screening alone has a limited impact on disease control, a conclusion that is consistent with the findings obtained for entry screening. This result further emphasizes the importance of combining screening measures with additional interventions, such as vaccination, to achieve more substantial reductions in disease transmission.

The control profiles with the bounds displayed in Table 5 indicate that, whether implemented alone or in combination with exit screening, vaccination should be maintained at its upper bound for approximately 37 weeks in Guinea, 39 weeks in Liberia, and 35 weeks in Sierra Leone. When exit screening is implemented as the sole intervention, it remains at its upper bound for approximately 20 weeks in Guinea, 39 weeks in Liberia, and 9 weeks in Sierra Leone. However, when combined with vaccination, the duration for which exit screening is maintained at its maximum level decreases to approximately 4.5 weeks, 38.5 weeks, and 4 weeks in Guinea, Liberia, and Sierra Leone, respectively, before gradually declining to its lower bound at the final time. These findings indicate that the addition of vaccination substantially reduces the effort required from exit screening while achieving more effective control of EVD transmission.

Along the lines of the numerical simulations presented in this section, we observe that the classical Euler-based optimal control schemes fail to be dynamically consistent with respect to the positivity and the boundedness of the solution, a shortcoming of classical schemes that has been extensively investigated in the literature (see [29, 31, 33]), and on which we gave

Entry screening	Guinea	Liberia	SL
$T_{\tau_i}^*$	32.5	18	2
$T_{\tau_i}^{**}$	40	40	40
Exit screening alone	Guinea	Liberia	SL
$T_{\eta_i}^*$	20	39	9
$T_{\eta_i}^{**}$	40	40	40
Exit screening when combined with vaccination	Guinea	Liberia	SL
$T_{\eta_i}^*$	4.5	38.5	4
$T_{\eta_i}^{**}$	40	40	40
Vaccination both alone and combined with exit screening	Guinea	Liberia	SL
$T_{\sigma_i}^*$	29	38	22
$T_{\sigma_i}^{**}$	40	40	40

Table 5: Values in weeks of times $T_{c_i}^*$ and $T_{c_i}^{**}$ for the incorporation of bounds per type of control, $c_i = \tau_i$ (entry screening), $c_i = \eta_i$ (exit screening) or $c_i = \sigma_i$ (vaccination).

above further explanations in terms of the unconditional stability and the absolute elementary stability of NSFD schemes for stiff or no stiff systems. Of great interest for optimal control problems in this work, numerical simulations suggest a new fact on the superiority of the NSFD schemes over the standard ones. That is, the classical Euler-based schemes annihilate the impact of the optimal control on the transmission of EVD, while the NSFD Euler-based schemes, whatever the time step, faithfully replicate its impact (see, for instance, Figure 7).

V. CONCLUSION

Since its first occurrence in 1976, the Ebola Virus Disease (EVD) has been a great public health burden to Sub-Saharan Africa and beyond due to its widespread and its mortality, ranging from 25% to 90%. Of the 30 recorded outbreaks, the most severe one started in 2013 in Guinea and moved to Liberia and Sierra Leone due to huge migrations of people from these neighboring countries.

In this paper, we extended the metapopulation model proposed in [10] by adding optimal entry screening and vaccination commands, both combined with the exit screening.

- We proved the existence and the uniqueness of the optimal controls and their associated state trajectories.
- From the computational analysis perspective, we designed and implemented nonstandard versions of the Forward-Backward Sweep Method (FBSM) for our optimal control models.

- In particular, numerical simulations brought to light a new fact that demonstrates the superiority of NSFD schemes over standard ones, namely the classical Euler-based schemes annihilate the impact of the optimal control on the transmission of EVD, while the NSFD Euler-based schemes faithfully replicate its impact. Additionally, they point out that, unlike screening measures, vaccination is highly effective in curbing the spread of EVD. The limited impact of screening can be attributed to migration patterns that do not rely on road travel. When these alternative modes of travel are taken into account, the results remain consistent, though the effect becomes more pronounced [35].

Some recommendations that arise from this work are as follows:

- To intensify the campaign for vaccination against Ebola (see [50] for existing vaccines).
- To manage travels and migrations between patches by combining exit and entry screenings to mitigate the EVD spread.

Our plan for future research is:

- To construct a model incorporating various cost-effective measures with the recently approved Ebanga drug for treating Ebola [51].
- To develop an Ebola patchy model that takes into account the residence place of individuals [16] and the time residents of one patch spent in the other. This is very important as the 2014–2016 outbreak occurred in neighboring countries.
- The development of a metapopulation model that considers the influence of the consumption of infected animals on the resurgence of EVD [12].

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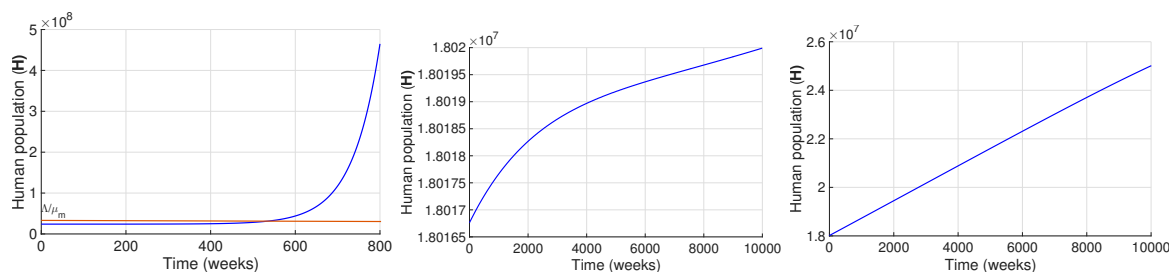


Fig. 2: Left figure: Dynamic inconsistency of *ode45* with regard to the conservation law (5). Middle and right figures: dynamic consistency of the NSFD scheme (12) (when $\tau = 0.2$ (middle figure)) and (19) (with $\eta = 0.2, \sigma = 0.01, \kappa = 0.0002$ (right picture)) to preserve the conservation Law (5). These figures are plotted with the initial conditions $S_1(0) = S_2(0) = S_3(0) = 4,000,000$ and the recruitment constant $\Lambda_i = 2000, \forall i = 1, 2, 3$. The carrying capacity of the total population is $\Lambda/\mu_m = 30,000,000$. The other parameters and initial conditions are as in Table 4 and Table 3, respectively.

AUTHOR CONTRIBUTIONS

The authors contributed equally to all aspects of to the work e.g. conceptualization, methodology development, analysis, numerical simulations, writing of manuscript, its final review, etc.

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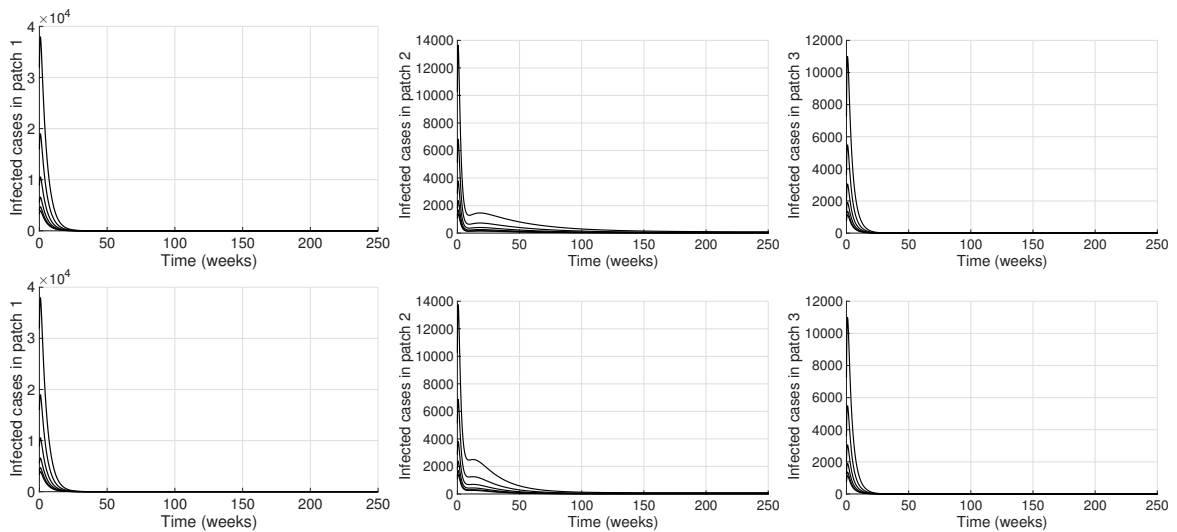


Fig. 3: Top row: Dynamic consistency of the forward NSFD scheme (12) with respect to the GAS of Model (2), with $\bar{\tau}_i = 0.2, i = 1, 2, 3, \beta_1 = \beta_3 = 0.1, \alpha_1 = \alpha_2 = 1.944$. The other parameters are as in Table 4. In the top row, $\mathcal{R}_c = 0.302$ and $\mathcal{T} = 0.8164 < 1$. Bottom row: Model behavior when $\mathcal{R}_c < 1 < \mathcal{T}$. These figures are plotted with $\beta_2 = 0.25$. The other parameters are as in the top row. In the bottom row, $\mathcal{R}_c = 0.4952$ and $\mathcal{T} = 1.1147$.

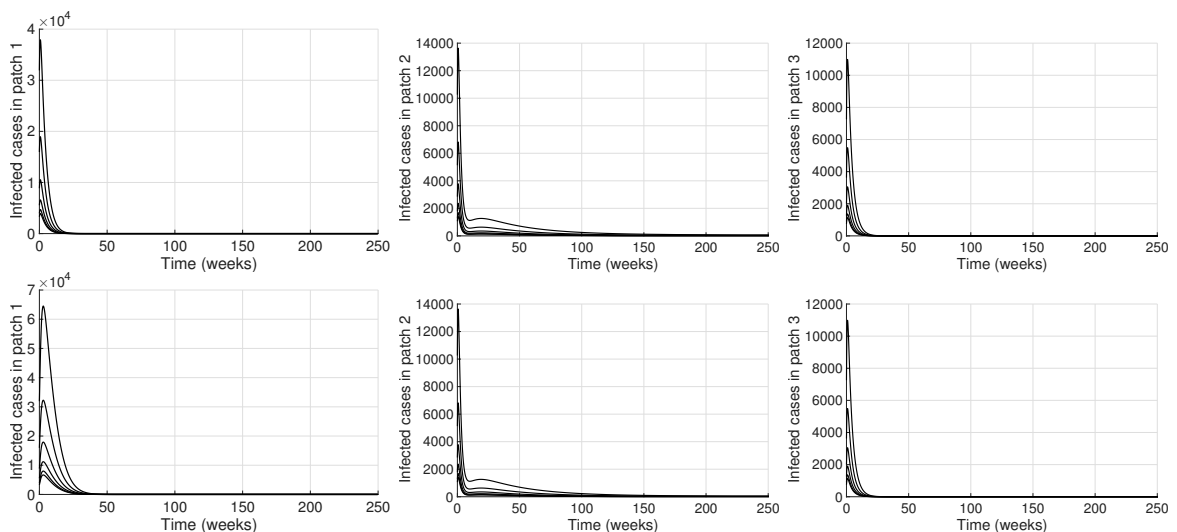


Fig. 4: Top row: Dynamic consistency of the forward NSFD scheme (19) with respect to the GAS of Model (14), with $\kappa = 0.2; \bar{\sigma} = 0.01; \eta = 0.2$. The other parameters are as Figure 3. In the top row, $\mathcal{R}_{c, \bar{\eta}, \bar{\sigma}} = \bar{\mathcal{T}}_{\bar{\eta}, \bar{\sigma}} = 0.302 < 1$. Bottom row: Model behavior when $\mathcal{R}_c < 1 = \bar{\mathcal{T}}_{\bar{\eta}, \bar{\sigma}}$. These figures are plotted with $\beta_1 = 0.3311$. The other parameters are as in the top row. In the bottom row, $\mathcal{R}_{c, \bar{\eta}, \bar{\sigma}} = 0.9999 < 1 = \bar{\mathcal{T}}_{\bar{\eta}, \bar{\sigma}}$.

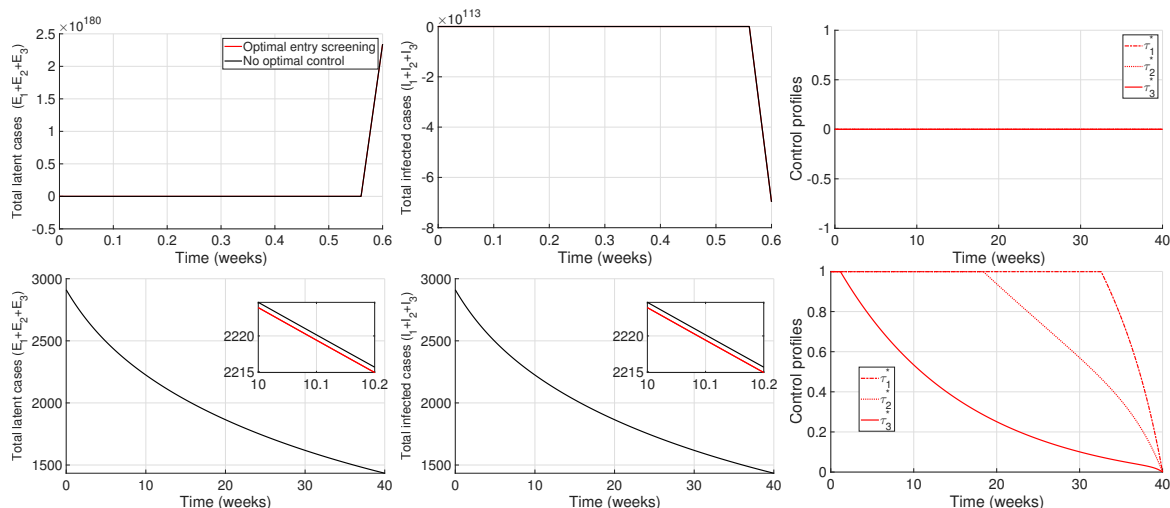


Fig. 5: Top three figures: Optimal entry screening problem solved with the classical Euler FBSM. Bottom three figures: Optimal entry screening problem solved with Euler-based NSFD-FBSM and with the parameters $\delta_3 = 40.5$, $w_1 = 200$; $w_2 = w_3 = 100$; $c_1 = c_3 = 200$; $c_2 = 20$; $q = 1000$; $h = 1$ for the state variables and $h = 0.005$ for adjoint variables. The other values of the parameters are in Table 4.

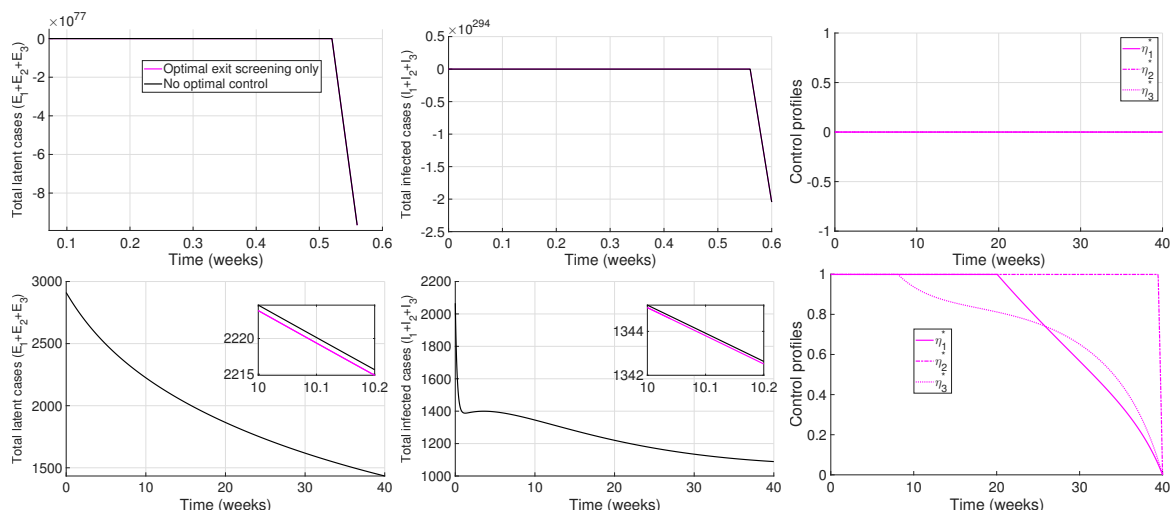


Fig. 6: Top three figures: Optimal exit screening problem solved with the classical Euler FBSM. Bottom three figures: Optimal exit screening problem solved with Euler-based NSFD-FBSM and with the parameters $\delta_3 = 40.5$, $w_1 = 200$; $w_2 = 1000$; $w_3 = 100$; $c_1 = 200$; $c_2 = 20$; $c_3 = 100$; $q = 1000$; $h = 1$ for the state variables and $h = 0.005$ for adjoint variables. The other values of the parameters are in Table 4.

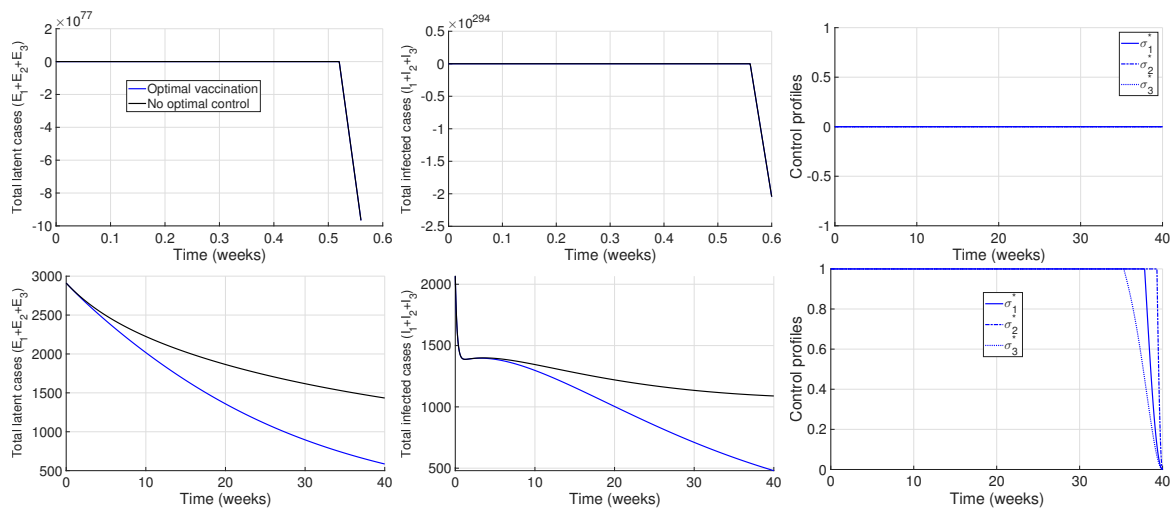


Fig. 7: Top three figures: Optimal vaccination problem solved with the classical Euler FBSM. Bottom three figures: Optimal vaccination problem solved with Euler-based NSFD-FBSM and with the parameters $\delta_3 = 40.5$; $w_1 = 200$; $w_2 = 1000$; $w_3 = 100$; $d_1 = d_2 = d_3 = 200$; $q = 1000$; $h = 1$ for the state variables and $h = 0.005$ for adjoint variables. The other values of the parameters are in Table 4.

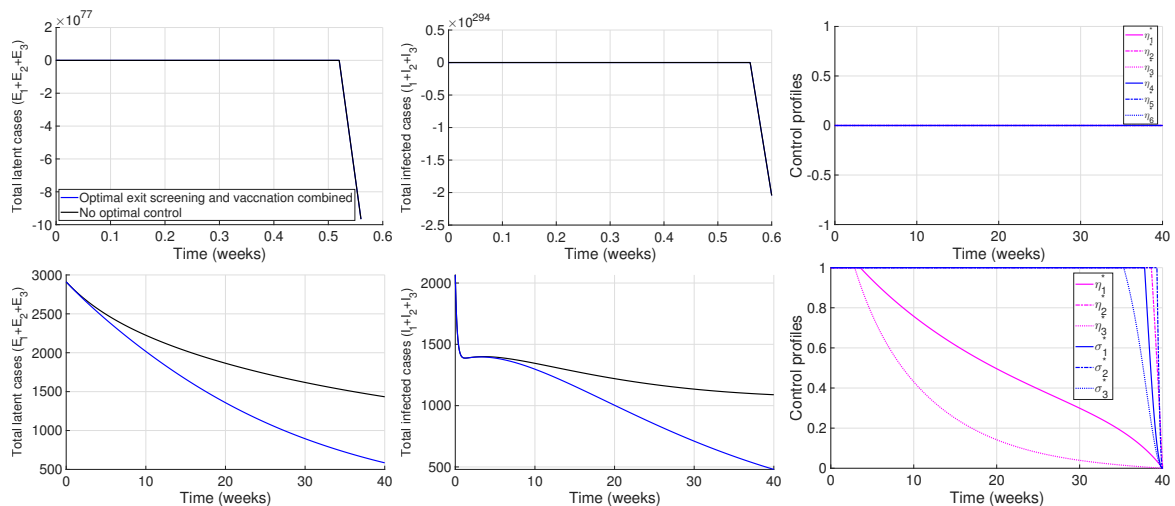


Fig. 8: Top three figures: Optimal exit screening and vaccination combined solved with the classical Euler FBSM. Bottom three figures: Optimal exit screening and vaccination combined solved with Euler-based NSFD-FBSM and with the parameters $\delta_3 = 40.5$; $w_1 = 200$; $w_2 = 1000$; $w_3 = 100$; $c_1 = 200$; $c_2 = 20$; $c_3 = 100$; $d_1 = d_2 = d_3 = 200$; $q = 1000$; $h = 1$ for the state variables and $h = 0.005$ for adjoint variables. The other values of the parameters are in Table 4.

VI. APPENDIX

Hamiltonian:

$$\begin{aligned}
 \mathcal{H} = & \sum_{i=1}^n (w_i E_i(t) + \frac{c_i}{2} \tau_i^2(t)) \\
 & + \sum_{k=1}^n z_{1k} \left(\Lambda_k - \beta_k \left(\frac{S_k I_k}{N_k} + \nu_k \frac{S_k D_k}{N_k} \right) - \mu_k S_k + \sum_{\substack{j=1 \\ j \neq k}}^n a_{kj} (1 - \eta_j^S) S_j - \sum_{\substack{j=1 \\ j \neq k}}^n a_{jk} S_k + \sum_{j=1}^n (1 - \phi_j) v_j \xi_{kj} Q_j \right) \\
 & + \sum_{k=1}^n z_{2k} \left(\beta_k \frac{S_k I_k + \nu_k S_k D_k}{N_k} - (\alpha_k + \mu_k) E_k + \sum_{\substack{j=1 \\ j \neq k}}^n a_{kj} (1 - \eta_j^E) (1 - \tau_k(t)) E_j - \sum_{\substack{j=1 \\ j \neq k}}^n a_{jk} E_k \right) \\
 & + \sum_{k=1}^n z_{3k} \left(\alpha_k E_k - (\mu_k + \delta_k + \gamma_k) I_k \right) + \sum_{k=1}^n z_{4k} \left((\mu_k + \delta_k) I_k - b_k D_k \right) \\
 & + \sum_{k=1}^n z_{5k} \left(\sum_{\substack{j=1 \\ j \neq k}}^n a_{jk} \eta_k^E E_k + \sum_{\substack{j=1 \\ j \neq k}}^n a_{kj} (1 - \eta_j^E) \tau_k(t) E_j + \sum_{\substack{j=1 \\ j \neq k}}^n a_{jk} \eta_k^S S_k - (\mu_k + \nu_k) Q_k \right) \\
 & + \sum_{k=1}^n z_{6k} \left(\phi_k v_k Q_k - (\mu_k + \psi_k + \theta_k) P_k \right) + \sum_{k=1}^n z_{7k} (\gamma_k I_k + \theta_k P_k - \mu_k R_k). \tag{20}
 \end{aligned}$$

Detailed system of ordinary differential equations for the adjoint variables, for $i = 1, \dots, n$,

$$\begin{aligned}
 \dot{z}_{1i}(t) = & -\frac{\partial \mathcal{H}}{\partial S_i} = \beta_i ((I_i + \nu_i D_i)/N_i) (z_{1i} - z_{2i}) + \mu_i z_{1i} + z_{1i} \sum_{\substack{j=1 \\ j \neq i}}^n a_{ji} \\
 & - z_{5i} \sum_{\substack{j=1 \\ j \neq i}}^n a_{ji} \eta_i^S - \sum_{\substack{j=1 \\ j \neq i}}^n z_{1j} a_{ji} (1 - \eta_i^S) + \left(v_i Q_i (z_{6i} - z_{1i}) \phi_i^2 \right) / \left(\sum_{p=1}^n E_p \right), \\
 \dot{z}_{2i}(t) = & -\frac{\partial \mathcal{H}}{\partial E_i} = -w_i + z_{2i} \left(\mu_i + \alpha_i + \sum_{\substack{j=1 \\ j \neq i}}^n a_{ji} \right) - z_{3i} \alpha_i - z_{5i} \sum_{\substack{j=1 \\ j \neq i}}^n a_{ji} \eta_i^E - \sum_{\substack{j=1 \\ j \neq i}}^n z_{5j} a_{ji} (1 - \eta_i^E) \tau_j(t) \\
 & - \sum_{\substack{j=1 \\ j \neq i}}^n z_{2j} a_{ji} (1 - \eta_i^E) (1 - \tau_j(t)) + \sum_{k,j=1}^n z_{1k} v_j \xi_{kj} Q_j S_j \phi_j^2 / \left(\sum_{p=1}^n E_p \right)^2 - \sum_{k=1}^n z_{6k} v_k Q_k S_k \phi_k^2 / \left(\sum_{p=1}^n E_p \right)^2, \\
 \dot{z}_{3i}(t) = & -\frac{\partial \mathcal{H}}{\partial I_i} = \frac{\beta_i S_i}{N_i} (z_{1i} - z_{2i}) + z_{3i} (\mu_i + \delta_i + \gamma_i) - z_{4i} (\mu_i + \delta_i) - z_{7i} \gamma_i, \\
 \dot{z}_{4i}(t) = & -\frac{\partial \mathcal{H}}{\partial D_i} = \frac{\beta_i \nu_i S_i}{N_i} (z_{1i} - z_{2i}) + z_{4i} b_i, \\
 \dot{z}_{5i}(t) = & -\frac{\partial \mathcal{H}}{\partial Q_i} = -z_{1i} (1 - \phi_i) v_i \xi_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^n z_{1j} (1 - \phi_i) v_i \xi_{ji} + z_{5i} (\mu_i + \nu_i) - z_{6i} \phi_i v_i, \\
 \dot{z}_{6i}(t) = & -\frac{\partial \mathcal{H}}{\partial P_i} = z_{6i} (\mu_i + \psi_i + \theta_i) - z_{7i} \theta_i, \\
 \dot{z}_{7i}(t) = & -\frac{\partial \mathcal{H}}{\partial R_i} = z_{7i} \mu_i. \tag{21}
 \end{aligned}$$

Detailed expression of the NSFD scheme (12) for Model (11):

$$\begin{aligned}
 \frac{S_i^{k+1,m+1} - S_i^{k,m+1}}{r} &= \Lambda_i - \lambda_i^{k,m+1} S_i^{k,m+1} - \mu_i S_i^{k,m+1} - \sum_{\substack{j=1 \\ j \neq i}}^n a_{ji} S_i^{k,m+1} \\
 &\quad + \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij} (1 - \eta_j^S) S_j^{k,m+1} + \sum_{j=1}^n (1 - \phi_j^{k,m+1}) v_j \xi_{ij} Q_j^{k,m+1}, \\
 \frac{E_i^{k+1,m+1} - E_i^{k,m+1}}{r} &= \lambda_i^{k,m+1} S_i^{k,m+1} - \mu_i E_i^{k,m+1} - \alpha_i E_i^{k,m+1} - \sum_{\substack{j=1 \\ j \neq i}}^n a_{ji} E_i^{k,m+1} \\
 &\quad + \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij} (1 - \eta_j^E) (1 - \tau_i^{k,m}) E_j^{k,m+1}, \\
 \frac{I_i^{k+1,m+1} - I_i^{k,m+1}}{r} &= \alpha_i E_i^{k,m+1} - (\mu_i + \delta_i + \gamma_i) I_i^{k,m+1}, \\
 \frac{D_i^{k+1,m+1} - D_i^{k,m+1}}{r} &= (\mu_i + \delta_i) I_i^{k,m+1} - b_i D_i^{k,m+1}, \\
 \frac{Q_i^{k+1,m+1} - Q_i^{k,m+1}}{r} &= \sum_{\substack{j=1 \\ j \neq i}}^n a_{ji} \eta_i^E E_i^{k,m+1} + \sum_{\substack{j=1 \\ j \neq i}}^n a_{ji} \eta_i^S S_i^{k,m+1} \\
 &\quad + \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij} (1 - \eta_j^E) \tau_i^{k,m} E_j^{k,m+1} - \mu_i Q_i^{k,m+1} - v_i Q_i^{k,m+1}, \\
 \frac{P_i^{k+1,m+1} - P_i^{k,m+1}}{r} &= \phi_i^{k,m+1} v_i Q_i^{k,m+1} - (\mu_i + \psi_i + \theta_i) P_i^{k,m+1}, \\
 \frac{R_i^{k+1,m+1} - R_i^{k,m+1}}{r} &= \gamma_i I_i^{k,m+1} + \theta_i P_i^{k,m+1} - \mu_i R_i^{k,m+1}.
 \end{aligned} \tag{22}$$

The classical scheme (13) for the adjoint system reads as:

$$\begin{aligned}
 \frac{z_{1i}^{k+1,m+1} - z_{1i}^{k,m+1}}{\Delta t} &= \lambda_i^{k+1,m+1} (z_{1i}^{k+1,m+1} - z_{2i}^{k+1,m+1}) \\
 &\quad + \mu_i z_{1i}^{k+1,m+1} + z_{1i}^{k+1,m+1} \sum_{\substack{j=1 \\ j \neq i}}^n a_{ji} - z_{5i}^{k+1,m+1} \sum_{j=1, j \neq i}^n a_{ji} \eta_i^S - \sum_{\substack{j=1 \\ j \neq i}}^n z_{1j}^{k+1,m+1} a_{ji} (1 - \eta_i^S) \\
 &\quad + \frac{v_i Q_i^{k+1,m+1} (z_{6i}^{k+1,m+1} - z_{1i}^{k+1,m+1}) (\phi_i^{k+1,m+1})^2}{\sum_{p=1}^n E_p^{k+1,m+1}}, \\
 \frac{z_{2i}^{k+1,m+1} - z_{2i}^{k,m+1}}{\Delta t} &= -w_i + z_{2i}^{k+1,m+1} \left(\mu_i + \alpha_i + \sum_{\substack{j=1 \\ j \neq i}}^n a_{ji} \right) - z_{3i}^{k+1,m+1} \alpha_i
 \end{aligned}$$

$$\begin{aligned}
 & - \sum_{\substack{j=1 \\ j \neq i}}^n z_{2j}^{k+1,m+1} a_{ji} (1 - \eta_i^E) (1 - \tau_j(t)^{k+1,m}) - z_{5i}^{k+1,m+1} \sum_{\substack{j=1 \\ j \neq i}}^n a_{ji} \eta_i^E - \sum_{\substack{j=1 \\ j \neq i}}^n z_{5j}^{k+1,m+1} a_{ji} (1 - \eta_i^E) \tau_j(t)^{k+1,m} \\
 & + \sum_{l,j=1}^n z_{1l}^{k+1,m+1} v_j \xi_{lj} Q_j^{k+1,m+1} \frac{S_j^{k+1,m+1} (\phi_j^{k+1,m+1})^2}{\left(\sum_{p=1}^n E_p^{k+1,m+1} \right)^2} - \sum_{l=1}^n z_{6l}^{k+1,m+1} v_l Q_l^{k+1,m+1} \frac{S_l^{k+1,m+1} (\phi_l^{k+1,m+1})^2}{\left(\sum_{p=1}^n E_p^{k+1,m+1} \right)^2}, \\
 & \frac{z_{3i}^{k+1,m+1} - z_{3i}^{k,m+1}}{\Delta t} = \frac{\beta_i S_i^{k+1,m+1}}{N_i^{k+1,m+1}} (z_{1i}^{k+1,m+1} - z_{2i}^{k+1,m+1}) + z_{3i}^{k+1,m+1} (\mu_i + \delta_i + \gamma_i) \\
 & \quad - z_{4i}^{k+1,m+1} (\mu_i + \delta_i) - z_{7i}^{k+1,m+1} \gamma_i, \\
 & \frac{z_{4i}^{k+1,m+1} - z_{4i}^{k,m+1}}{\Delta t} = \frac{\beta_i^{k+1,m+1} \nu_i S_i^{k+1,m+1}}{N_i^{k+1,m+1}} (z_{1i}^{k+1,m+1} - z_{2i}^{k+1,m+1}) + z_{4i}^{k+1,m+1} b_i, \\
 & \frac{z_{5i}^{k+1,m+1} - z_{5i}^{k,m+1}}{\Delta t} = -z_{1i}^{k+1,m+1} (1 - \phi_i^{k+1,m+1}) v_i \xi_{ii} + z_{5i}^{k+1,m+1} (\mu_i + v_i) \\
 & \quad - \sum_{\substack{j=1 \\ j \neq i}}^n z_{1j}^{k+1,m+1} (1 - \phi_i^{k+1,m+1}) v_i \xi_{ji} - z_{6i}^{k+1,m+1} \phi_i^{k+1,m+1} v_i, \\
 & \frac{z_{6i}^{k+1,m+1} - z_{6i}^{k,m+1}}{\Delta t} = z_{6i}^{k+1,m+1} (\mu_i + \psi_i + \theta_i) - z_{7i}^{k+1,m+1} \theta_i, \\
 & \frac{z_{7i}^{k+1,m+1} - z_{7i}^{k,m+1}}{\Delta t} = z_{7i}^{k+1,m+1} \mu_i.
 \end{aligned} \tag{23}$$

The detailed expression of the state variables of the NSFD-FBSM, in Step 2, for the System (14) is, for all $i = 1, \dots, n$,

$$\begin{aligned}
 \frac{S_i^{k+1,m+1} - S_i^{k,m+1}}{r} &= \Lambda_i - \lambda_i^{k,m+1} S_i^{k,m+1} - \mu_i S_i^{k,m+1} \\
 &\quad - \sum_{\substack{j=1 \\ j \neq i}}^n a_{ji} S_i^{k,m+1} + \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij} S_j^{k,m+1} - \kappa \sigma_i^{k,m} S_i^{k,m+1}, \\
 \frac{E_i^{k+1,m+1} - E_i^{k,m+1}}{r} &= \lambda_i^{k,m+1} S_i^{k,m+1} - \mu_i E_i^{k,m+1} - \alpha_i E_i^{k,m+1} \\
 &\quad - \sum_{\substack{j=1 \\ j \neq i}}^n a_{ji} E_i^{k,m+1} + \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij} (1 - \eta_j^{k,m}) E_j^{k,m+1}, \\
 \frac{I_i^{k+1,m+1} - I_i^{k,m+1}}{r} &= \alpha_i E_i^{k,m+1} - (\mu_i + \delta_i + \gamma_i) I_i^{k,m+1}, \\
 \frac{D_i^{k+1,m+1} - D_i^{k,m+1}}{r} &= (\mu_i + \delta_i) I_i^{k,m+1} - b_i D_i^{k,m+1}, \\
 \frac{Q_i^{k+1,m+1} - Q_i^{k,m+1}}{r} &= \sum_{j=1}^n a_{ji} \eta_i^{k,m} E_i^{k,m+1} - \mu_i Q_i^{k,m+1} - v_i Q_i^{k,m+1}, \\
 \frac{P_i^{k+1,m+1} - P_i^{k,m+1}}{r} &= v_i Q_i^{k,m+1} - (\mu_i + \psi_i + \theta_i) P_i^{k,m+1}, \\
 \frac{R_i^{k+1,m+1} - R_i^{k,m+1}}{r} &= \kappa \sigma_i^{k,m} S_i^{k,m+1} + \gamma_i I_i^{k,m+1} + \theta_i P_i^{k,m+1} - \mu_i R_i^{k,m+1}.
 \end{aligned} \tag{24}$$

Finally, expressions of σ^* , η^* and discrete (co-) state variables. In particular, the detailed expression of the Hamiltonian $\mathcal{H}$ defined in (17) is

$$\begin{aligned} \mathcal{H} = & \sum_{i=1}^n w_i E_i(t) + \frac{d_i}{2} \sigma_i^2(t) + \frac{c_i}{2} \eta_i^2(t) + z_{7i}(\sigma_i(t) \kappa S_i(t) + \gamma_i I_i + \theta_i P_i - \mu_i R_i) \\ & + z_{1i} \left(\Lambda_i - \beta_i (S_i I_i + \nu_i S_i D_i) / N_i - \mu_i S_i + \sum_{j=1}^n a_{ij} S_j - \sum_{j=1}^n a_{ji} S_i - \kappa \sigma_i(t) S_i \right) \\ & + z_{2i} \left(\beta_i (S_i I_i + \nu_i S_i D_i) / N_i - \mu_i E_i - \alpha_i E_i + \sum_{j=1}^n a_{ij} (1 - \eta_j(t)) E_j - \sum_{j=1}^n a_{ji} E_i \right) \\ & + z_{3i} (\alpha_i E_i - (\mu_i + \delta_i + \gamma_i) I_i) + z_{4i} ((\mu_i + \delta_i) I_i - b_i D_i) \\ & + z_{5i} \left(\sum_{j=1}^n a_{ji} \eta_j(t) E_i - (\mu_i + v_i) Q_i \right) + z_{6i} (v_i Q_i - (\mu_i + \psi_i + \theta_i) P_i). \end{aligned} \quad (25)$$

Simple computations show that the adjoint variables satisfy the system

$$\begin{aligned} \dot{z}_{1i}(t) &= -\frac{\partial \mathcal{H}}{\partial S_i} = \beta_i ((I_i + \nu_i D_i) / N_i) (z_{1i} - z_{2i}) + \mu_i z_{1i} + z_{1i} \sum_{j=1}^n a_{ji} + \kappa \sigma_i(t) z_{1i} - z_{7i} \sigma_i(t) \kappa - \sum_{\substack{j=1 \\ j \neq i}}^n z_{1j} a_{ji}, \\ \dot{z}_{2i}(t) &= -\frac{\partial \mathcal{H}}{\partial E_i} = -w_i + z_{2i} (\mu_i + \alpha_i + \sum_{j=1}^n a_{ji}) - z_{3i} \alpha_i - z_{5i} \sum_{j=1}^n a_{ji} \eta_j(t) - \sum_{\substack{j=1 \\ j \neq i}}^n z_{2j} a_{ji} (1 - \eta_j(t)), \\ \dot{z}_{3i}(t) &= -\frac{\partial \mathcal{H}}{\partial I_i} = ((\beta_i S_i) / N_i) (z_{1i} - z_{2i}) + z_{3i} (\mu_i + \delta_i + \gamma_i) - z_{4i} (\mu_i + \delta_i) - z_{7i} \gamma_i, \\ \dot{z}_{4i}(t) &= -\frac{\partial \mathcal{H}}{\partial D_i} = ((\beta_i \nu_i S_i) / N_i) (z_{1i} - z_{2i}) + z_{4i} b_i, \\ \dot{z}_{5i}(t) &= -\frac{\partial \mathcal{H}}{\partial Q_i} = z_{5i} (\mu_i + v_i) - z_{6i} v_i, \\ \dot{z}_{6i}(t) &= -\frac{\partial \mathcal{H}}{\partial P_i} = z_{6i} (\mu_i + \psi_i + \theta_i) - z_{7i} \theta_i, \\ \dot{z}_{7i}(t) &= -\frac{\partial \mathcal{H}}{\partial R_i} = z_{7i} \mu_i. \end{aligned} \quad (26)$$

Again, simple computations lead to

$$\frac{\partial \mathcal{H}}{\partial \sigma_i} = 0 \quad \text{at } \sigma_i = \sigma_i^*, \quad \frac{\partial \mathcal{H}}{\partial \eta_i} = 0 \quad \text{at } \eta_i = \eta_i^*.$$

From the bounds of the continuous optimal controls, we can take

$$\eta_i^*(t) = \begin{cases} 1, & \text{if } 0 \leq t < T_{\eta_i}^* \\ \frac{1}{c_i} \sum_{\substack{j=1 \\ j \neq i}}^n a_{ji} (z_{2j} - z_{5i}) E_i, & \text{if } T_{\eta_i}^* \leq t \leq T_{\eta_i}^{**} \\ 0, & \text{if } T_{\eta_i}^{**} < t \leq T \end{cases}$$

$$\sigma_i^*(t) = \begin{cases} 1, & \text{if } 0 \leq t < T_{\sigma_i}^* \\ \frac{S_i \kappa (z_{1i} - z_{7i})}{d_i}, & \text{if } T_{\sigma_i}^* \leq t \leq T_{\sigma_i}^{**} \\ 0, & \text{if } T_{\sigma_i}^{**} < t \leq T \end{cases}$$

This completes the proof of Theorem 6.