

Size structured population models

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Keywords: size structured populations, McKendrick model, growth-fragmentation-coagulation equations

Many living organisms, from bacteria, through phytoplankton, to elephants, have a tendency to form groups that can grow in size by demographic processes occurring inside each group, but also due to amalgamation and/or fragmentation of the groups, [6]. Similar processes are also important in engineering (polymerization/depolymerization), or planetary science (formation and destruction of planetismals from cosmic dust).

We are concerned with describing such processes using the number $u_i(t)$ of groups of size i at time t if i is a discrete variable, or a suitable density $u(x, t)$ of groups of size x if x is a continuous variable.

The modelling of such processes goes back to the work of Smoluchowski, [7] and Müller [5]. In our case, the classical fragmentation-coagulation equations are complemented by the terms describing demographical process, such as McKendrick-von Foerster renewal model. In the continuous case we study, [1, 3],

$$\begin{aligned}\partial_t u(x, t) &= -\partial_x(r(x)u(x, t)) - \mu(x)u(x, t) \\ &\quad - a(x)u(x, t) - \int_x^\infty a(y)b(x|y)u(y, t)dy \\ &\quad + u(x, t) \int_0^\infty k(x, y)u(y, t)dy \\ &\quad + \frac{1}{2} \int_0^x k(x-y, y)u(x-y, t)u(y, t)dy, \quad x > 0, \\ r(0)u(0, t) &= \int_0^\infty \beta(y)u(y, t)dy, \\ u(x, 0) &= u_0(x),\end{aligned}\tag{1}$$

where r is the size dependent growth rate of clusters, μ is the rate of the clusters removal, a is the rate of splitting, the fragmentation kernel b is the average number of clusters generated by fragmentation of a cluster of size y , k is the

amalgamation rate and β describes the rate at which newborns detach from the clusters.

In many cases it is unreasonable to assume continuous size distribution. If we discretize (1), we obtain (retaining the meaning of the coefficients)

$$\begin{aligned} \frac{du_i(t)}{dt} = & r_{i-1}u_{i-1}(t) - r_iu_i(t) - \mu_iu_i(t) + \frac{1}{2} \sum_{j=1}^{i-1} k_{i-j,j}u_{i-j}(t)u_j(t) \\ & - \sum_{j=1}^{\infty} k_{i,j}u_i(t)u_j(t) - a_iu_i(t) + \sum_{j=i+1}^{\infty} a_jb_{i,j}u_j(t), \quad i \geq 1, \end{aligned} \quad (2)$$

and the boundary equation for the class of newborns $u_0(t)$,

$$r_0u_0(t) = \sum_{j=1}^{\infty} \beta_ju_j(t), \quad (3)$$

unlike in the continuous case, can be inserted into (2).

The pure fragmentation semigroup was proved to be analytic for a large class of fragmentation kernels in [2, 4]; it is also compact in the discrete case. The main aim of this lecture is to show that these results can be extended to (2) and to apply them to analyze the long term behaviour of solutions to (2).

Acknowledgement. The presented results have been obtained in collaboration with W. Lamb (Strathclyde), S. Shindin and L. Joel (UKZN).

References

- [1] Azmy S Ackleh and Ben G Fitzpatrick. Modeling aggregation and growth processes in an algal population model: analysis and computations. *Journal of Mathematical Biology*, 35(4):480–502, 1997.
- [2] Jacek Banasiak. Global classical solutions of coagulation-fragmentation equations with unbounded coagulation rates. *Nonlinear Anal. Real World Appl.*, 13(1):91–105, 2012.
- [3] Jacek Banasiak and Wilson Lamb. Coagulation, fragmentation and growth processes in a size structured population. *Discrete Contin. Dyn. Syst. Ser. B*, 11(3):563–585, 2009.
- [4] Jacek Banasiak and Wilson Lamb. Global strict solutions to continuous coagulation-fragmentation equations with strong fragmentation. *Proc. Roy. Soc. Edinburgh Sect. A*, 141(3):465–480, 2011.

- [5] Hans Müller. Zur allgemeinen theorie ser raschen koagulation. *Fortschrittsberichte über Kolloide und Polymere*, 27(6):223–250, 1928.
- [6] Akira Okubo and Simon A. Levin. *Diffusion and ecological problems: modern perspectives*, volume 14 of *Interdisciplinary Applied Mathematics*. Springer-Verlag, New York, second edition, 2001.
- [7] M. V. Smoluchowski. Versuch einer mathematischen theorie der koagulation-skinetik kolloider lösungen. *Zeitschrift fuer physikalische Chemie*, 92:129 – 168, 2010.