Biomath Communications

www.biomathforum.org/biomath/index.php/conference

On the Uniform Stability of Impulsive Lotka-Volterra Type Systems with Supremums

Ivanka Stamova[1], Gani Stamov[2]
[1] Department of Mathematics, University of Texas at San Antonio, San Antonio, Texas, USA
E-mail: Ivanka.Stamova@utsa.edu
[2]Department of Mathematics, Technical University of Sofia, Sliven, Bulgaria
E-mail: gstamov@abv.bg

Absract: In this paper we propose an impulsive $n-$ species Lotka-Volterra model with supremums. By using Lyapunov method we give sufficient conditions for uniform stability and uniform asymptotic stability of the positive states.

AMS(MOS) Subject Classification: 34K13, 34D05, 34A37

Keywords: Lyapunov method, Stability, Impulsive perturbations, Lotka-Volterra competitive systems, Supremums

1 Introduction

Lotka-Volterra and related systems of differential equations have been extensively studied in the literature. See, for example, [1, 9, 21].

Citation: Ivanka Stamova, Gani Stamov, On the Uniform Stability of Impulsive Lotka-Volterra Type Systems with Supremums,
http://dx.doi.org/10.11145/bmc.2017.12.247

On the other hand, impulsive biological systems have gained increasing popularity during the last few decades. Impulsive conditions have been incorporated into biological models by many researchers to represent abrupt changes at certain moments of time. In term of the mathematical treatment, the presence of pulses gives the system a mixed nature, both continuous and discrete [4, 5, 10, 19]. For example, some impulsive differential equations have been recently introduced in population dynamics, such as vaccination [7], population growth models [2, 3, 14, 16], chemotherapeutic treatment of disease [11], chemostat [20], pollution [13], the tumor-normal cell interaction [8], etc.

Also, in the recent years, much attention has been paid to study impulsive Lotka-Volterra models. Recently, Li et al. [12] investigated the existence of positive periodic solutions of a two-dimensional Lotka-Volterra impulsive system with infinitely distributed delay, Liu et al. [15] have formulated a two-species Lotka-Volterra impulsive delay model with periodic coefficients and interesting results about permanence and extinction were obtained, also, theorems for the existence of semitrivial periodic solutions and the stability results using a modification of Lyapunov's first method are proved in [15]; some stability results for impulsive n-species Lotka-Volterra models are obtained in [19]; theorems for the existence of almost periodic solutions are given in [2, 17].

In mathematical simulation in various important branches of control theory, pharmacokinetics, economics, etc., one has to analyze the influence of both the maximum of the function investigated and its impulsive changes. Thus, for instance, if the concentration of the medicinal substance in the blood plasma has to be controlled at a venous injection of the medical substance, one has to take into account together with it, in view of the optimal therapy, the maximum of this concentration too. Similar problems appear in many fields of science and technology [18]. An adequate mathematical apparatus for simulation of such processes seems to be impulsive differential equations with supremums. To the best of our knowledge, there are no results considering the stability of Lotka-Volterra systems with supremums, which is important from a theoretical point of view as well as applications, and is also a challenging problem.

In this paper, we investigate the following n-species Lotka-Volterra type impulsive system with supremums

$$\begin{cases} \dot{x}_i(t) = x_i(t)\left[r_i(t) - a_{ii}(t)x_i(t) - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}(t) \sup_{s\in[t-\tau,t]} x_j(s)\right],\ t \neq t_k, \\ x_i(t_k^+) = x_i(t_k) + I_{ik}(x_i(t_k)),\ k = 1, 2, ..., \end{cases} \tag{1.1}$$

where $i = 1, ..., n$, $n \geq 2$; $t \geq 0$; $x_i(t)$ represents the density of species i at the moment t; $r_i(t)$ is the reproduction rate function; and $a_{ij}(t)$ are functions which describe the effect of the j -th population upon the i -th population; $a_{ij} \in C[[0,\infty),[0,\infty)]$; $r_i \in C[[0,\infty), R]$; I_{ik} : $[0,\infty) \to R$; $0 < t_1 < t_2 < ... < t_k < ...$ are fixed impulsive points and $\lim_{k\to\infty} t_k = \infty$.

In mathematical ecology, system (1.1) denotes a model of the dynamics of an n-species system in which each individual competes with all the others for a common resource and the intra-species competition depends on the maximum values of the population densities $x_j(s)$ $(j = 1, ..., n,\ j \neq i$, over the past time interval $[t-\tau, t]$, where $\tau > 0$ is a constant. The growth functions r_i are not necessarily positive, since the environment fluctuates randomly; in bad conditions, r_i may be negative. If at a certain time, biotic and antropogeneous factors act on the population "momentarily", then the population number varies by jumps. The numbers $x_i(t_k)$ and $x_i(t_k^+)$ are, respectively, the population densities of species i before and after impulse perturbation at the moment t_k; and I_{ik} are functions which characterize the magnitude of the impulse effect on the species i at the moments t_k.

In this paper, we derive sufficient conditions for uniform stability and uniform asymptotic stability of the positive solutions of system (1.1). The paper is organized as follows. In Section 2, we give some preliminaries and main definitions. In Section 3 we study the persistence of positive solutions of system (1.1). In Section 4, we investigate uniform stability and uniform asymptotic stability of positive solutions. By means of piecewise continuous Lyapunov method, sufficient conditions are obtained. Finally, in Section 5 two examples are given to illustrate the effectiveness of the results obtained. We show that by means of appropriate impulsive perturbations we can control the system's population dynamics. Some known results are improved and generalized.

2 Preliminaries

Let R^n denote the n-dimensional Euclidean space, and let $||x|| = |x_1| + ... + |x_n|$ be the norm of $x \in R^n$, $R_+ = [0, \infty)$.

Let $\phi \in CB[[-\tau, 0], R^n]$, where $CB[[-\tau, 0], R^n] = \{\sigma : [-\tau, 0] \to R^n$, $\sigma(t)$ continuous and bounded on $[-\tau, 0]$ $\}$, $\phi = col(\phi_1, \phi_2, ..., \phi_n)$. We denote by $x(t) = x(t; 0, \phi) = col(x_1(t; 0, \phi), x_2(t; 0, \phi), ..., x_n(t; 0, \phi))$ the solution of system (1.1), satisfying the initial conditions

$$\begin{cases} x_i(s; 0, \phi) = \phi_i(s), \; s \in [-\tau, 0], \\ x_i(0^+; 0, \phi) = \phi_i(0), \; i = 1, ..., n, \end{cases} \tag{2.1}$$

and by $J^+(0, \phi)$ - the maximal interval of type $[0, \beta)$ in which the solution $x(t; 0, \phi)$ is defined.

Let $||\phi||_\tau = \max_{s \in [-\tau, 0]} ||\phi(s)||$ be the norm of the function $\phi \in CB[[-\tau, 0], R^n]$.

We note that [19] the solution $x(t) = x(t; 0, \phi)$ of problem (1.1)-(2.1) is piecewise continuous function in $[0, \infty)$ with points of discontinuity of the first kind at t_k $(k = 1, 2, ...)$ at which it is left continuous, i.e. the following relations are satisfied:

$$x_i(t_k^-) = x_i(t_k), \quad k = 1, 2, ...,$$

$$x_i(t_k^+) = x_i(t_k) + I_{ik}(x_i(t_k)), \; t_k \in (0, \infty),$$

$i = 1, ..., n$.

We introduce the following assumptions:

A1. $r_i \in C[R_+, R]$, $i = 1, 2, ..., n$.

A2. $a_{ij} \in C[R_+, R_+]$, $i, j = 1, 2, ..., n$.

A3. $0 < t_1 < t_2 < ...$ and $\lim\limits_{k\to\infty} t_k = \infty$.
A4. $I_{ik} \in C[R_+, R]$, $i = 1,\ 2,\ ...,\ n$, $k = 1, 2, ...$.
A5. $x_i + I_{ik}(x_i) \geq 0$ for $x_i \in R_+$, $i = 1,\ 2,\ ...,\ n$, $k = 1, 2, ...$.

Given a continuous function $g(t)$ which is defined on J, $J \subseteq R$, we set

$$g^L = \inf_{t\in J} g(t), \quad g^M = \sup_{t\in J} g(t).$$

3 Permanence

In the proofs of the main theorems, we shall use the following lemmas.

Lemma 3.1. *Let the assumptions* A1-A4 *hold.*
Then $J^+(0, \phi) = [0, \infty)$.

Proof. Since the conditions A1 and A2 hold, then from the existence theorem for the corresponding system without impulses [9, 21], it follows that the solution $x(t) = x(t; 0, \phi)$ of problem (1.1)-(2.1) is defined on $[0, t_1] \cup (t_k, t_{k+1}]$, $k = 1, 2, ...$. From conditions A3 and A4, we conclude that it is continuable for $t \geq 0$.

Lemma 3.2. *Assume that:*
1. *Conditions* A1-A5 *hold.*
2. $x(t) = x(t; 0, \phi) = col(x_1(t; 0, \phi), x_2(t; 0, \phi), ..., x_n(t; 0, \phi))$ *is a solution of* (1.1)-(2.1) *such that*

$$x_i(s) = \phi_i(s) \geq 0, \quad \sup \phi_i(s) < \infty, \quad \phi_i(0) > 0, \ \ 1 \leq i \leq n.$$

Then $x_i(t) > 0, \quad 1 \leq i \leq n, \quad t \in [0, \infty)$.

Proof. Since $\phi_i(0) > 0$, the condition A5 holds, and the solution of (1.1) is defined by

$$x_i(t) = \phi_i(0) \exp\left\{\int_0^t \left[r_i(s) - a_{ii}(s)x_i(s) - \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}(s) \sup_{s\in[t-\tau,t]} x_j(s)\right] ds\right\}, \qquad t \in [0, t_1],$$

$$x_i(t)=x_i(t_k^+)\exp\left\{\int_{t_k}^{t}\left[r_i(s)-a_{ii}(s)x_i(s)-\sum_{\substack{j=1\\ j\neq i}}^{n}a_{ij}(s)\sup_{s\in[t-\tau,t]}x_j(s)\right]ds\right\},$$
$$t\in(t_k,t_{k+1}],$$
$$x_i(t_k^+)=x_i(t_k)+I_{ik}(x_i(t_k)),\quad i=1,2,...,n,\quad k=1,2,...,$$
then the solution of (1.1) is positive for $t\in[0,\infty)$.

Definition 3.1. The solution $\overline{x}:[0,\infty)\to R^n$ of the system (1.1) is said to be a *maximal solution* if for any other solution $x:[0,\infty)\to R^n$ of the system (1.1) the inequality $x(t)\le\overline{x}(t)$ holds for $t\in[0,\infty)$.

The minimal solution $\underline{x}(t)$ of the system (1.1) can be defined analogously by reversing the above inequality.

Lemma 3.3. *Assume that:*

1. *Conditions of Lemma* 3.2 *hold.*

2. *The function $U_i(t)\ge 0$ is the maximal solution of the logistic system*
$$\begin{cases} \dot U_i(t)=U_i(t)\left[|r_i^M|-a_{ii}^L U_i(t)\right],\ t\neq t_k,\\ U_i(t_k^+)=U_i(t_k)+I_{ik}^M, \end{cases}$$
where $I_{ik}^M=\max\{I_{ik}(U_i(t_k))\}$ for $1\le i\le n$ and $k=1,2,....$

3. *The function $V_i(t)\ge 0$ is the minimal solution of the system*
$$\begin{cases} \dot V_i(t)=V_i(t)\left[r_i^L-a_{ii}^M V_i(t)-\sum\limits_{\substack{j=1\\ j\neq i}}^{n}a_{ij}^M\sup\limits_{t-\tau\le s\le t}U_j(s)\right],\ t\neq t_k,\\ V_i(t_k^+)=V_i(t_k)+I_{ik}^L, \end{cases}$$
where $I_{ik}^L=\min\{I_{ik}(V_i(t_k))\}$ for $1\le i\le n$ and $k=1,2,....$

4. $0\le V_i(0^+)\le\phi_i(0)\le U_i(0^+)$, $1\le i\le n$.

Then
$$V_i(t)\le x_i(t)\le U_i(t),\quad 1\le i\le n,\quad t\in[0,\infty).\qquad(3.1)$$

Proof. Since all conditions of Lemma 3.2 are satisfied, the domain $\{col(x_1,\ x_2,\ ...,x_n):\ x_i>0, i=1,2,...,n\}$ is positive invariant with respect to system (1.1).

¿From (1.1), for $i = 1, 2, ..., n$, we have

$$\begin{cases} \dot{x}_i(t) \le x_i(t) \left[|r_i^M| - a_{ii}^L x_i(t) \right], \ t \ne t_k, \\ x_i(t_k^+) \le x_i(t_k) + I_{ik}^M, \ k = 1, 2, ..., \end{cases}$$

and

$$\begin{cases} \dot{x}_i(t) \ge x_i(t) \left[r_i^L - a_{ii}^M x_i(t) - \sum_{\substack{j=1 \\ j \ne i}}^n a_{ij}^M \sup_{t-\tau \le s \le t} x_j(s) \right], \ t \ne t_k, \\ x_i(t_k^+) \ge x_i(t_k) + I_{ik}^L, \ k = 1, 2, ... \end{cases}$$

Then from the differential inequalities for piecewise continuous functions $V_i(t)$, $U_i(t)$ and $x_i(t)$ ([10]), we obtain that (3.1) is valid for $t \in [0, \infty)$ and $1 \le i \le n$.

Lemma 3.4. *Let the conditions of Lemma 3.2 hold and*

$$r_i^L \ge \sum_{\substack{j=1 \\ j \ne i}}^n \frac{a_{ij}^M r_j^M}{a_{ii}^L}, \quad i, j = 1, 2, ..., n.$$

Then for all $t \in [0, t_1] \cup (t_k, t_{k+1}]$, $k = 1, 2, ...$ and $1 \le i \le n$, we have

$$\alpha_i \le x_i(t) \le \beta_i, \tag{3.2}$$

where

$$\alpha_i = \frac{r_i^L - \sum_{\substack{j=1 \\ j \ne i}}^n \frac{a_{ij}^M r_j^M}{a_{ii}^L}}{a_{ii}^M}, \quad \beta_i = \frac{|r_i^M|}{a_{ii}^L}.$$

If, in addition, the functions I_{ik} are such that

$$\alpha_i \le x_i + I_{ik}(x_i) \le \beta_i$$

for $x_i \in R_+$, $i = 1, 2, ..., n$, $k = 1, 2, ...$, then the inequalities (3.2) *are valid for all $t \in [0, \infty)$ and $1 \le i \le n$.*

Proof. From Lemma 3.3, we have that (3.1) holds for $t \in [0, \infty)$ and $1 \le i \le n$. We shall prove that there exist positive constants α_i and β_i such that

$$\alpha_i \le V_i(t) \le U_i(t) \le \beta_i \tag{3.3}$$

for all $t \in [0, t_1] \cup (t_k, t_{k+1}]$, $k = 1, 2, ...$ and $1 \le i \le n$.

First, we shall prove that

$$U_i(t) \le \beta_i \tag{3.4}$$

for all $t \in [0, t_1] \cup (t_k, t_{k+1}]$, $k = 1, 2, ...$ and $1 \le i \le n$.

If $t \in [0, \infty)$, $t \neq t_k$ and for some i, $i = 1, 2, ..., n$, $U_i(t) > \beta_i$, then for $t \in [0, t_1] \cup (t_k, t_{k+1}]$, $k = 1, 2, ...$, we will have

$$\dot{U}_i(t) < U_i(t) \left[|r_i^M| - a_{ii}^L U_i(t) \right] < 0.$$

This proves that (3.4) holds for all $t \in [0, t_1] \cup (t_k, t_{k+1}]$, $k = 1, 2, ...$ and $i = 1, 2, ..., n$, as long as $U_i(t)$ is defined.

The inequality $\alpha_i \le V_i(t)$ is proved similarly.

Hence, the inequalities (3.3) are valid for all $t \in [0, t_1] \cup (t_k, t_{k+1}]$, $k = 1, 2, ...$ and $1 \le i \le n$.

It is now clear that if, in addition, the functions I_{ik} are such that $\alpha_i \le x_i(t_k) + I_{ik}(x_i(t_k)) \le \beta_i$ for $x_i \in R_+$, $i = 1, 2, ..., n$, $k = 1, 2, ...$, then inequalities (3.3) are valid for all $i = 1, 2, ..., n$ and $t \in [0, \infty)$.

Corollary 3.1. *Let the conditions of Lemma* 3.4 *hold, and the functions* I_{ik} *are such that*

$$\alpha_i \le x_i + I_{ik}(x_i) \le \beta_i \quad for \quad x_i \in R_+, \;\; i = 1, 2, ..., n, \;\; k = 1, 2, ...$$

Then there exist positive constants m *and* M, m, $M < \infty$ *such that*

$$m \le x_i(t) \le M, \quad t \in [0, \infty). \tag{3.5}$$

4 Uniform stability

Let $\varphi \in CB[[-\tau, 0], R^n]$, $\varphi = col(\varphi_1, \varphi_2, ..., \varphi_n)$ and $x^*(t) = x^*(t; 0, \varphi) = col(x_1^*(t; 0, \varphi), x_2^*(t; 0, \varphi), ..., x_n^*(t; 0, \varphi))$ be a strictly positive (componentwise) solution of system (1.1), satisfying the initial conditions

$$\begin{cases} x_i^*(s;0,\varphi) = \varphi_i(s), \ s \in [-\tau, 0], \\ x_i^*(0^+;0,\varphi) = \varphi_i(0), \quad i = 1, 2, ..., n. \end{cases}$$

Next, we suppose that

$$\varphi_i(s) \geq 0, \quad \sup \varphi_i(s) < \infty, \quad \varphi_i(0) > 0,$$

$$\phi_i(s) \geq 0, \quad \sup \phi_i(s) < \infty, \quad \phi_i(0) > 0, \quad i = 1, 2, ..., n.$$

In this paper we will use the following definitions for uniform stability and asymptotic stability of the solutions of (1.1)

Definition 4.1. The solution $x^*(t) = x^*(t;0,\varphi)$ of (1.1) is said to be:

(a) *uniformly stable*, if for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $||\phi - \varphi||_\tau < \delta$ implies $||x(t) - x^*(t)|| < \varepsilon$ for all $t \geq 0$;

(b) *uniformly asymptotically stable*, if it is uniformly stable and

$$\lim_{t \to \infty} ||x(t) - x^*(t)|| = 0.$$

Theorem 4.1. *Assume that:*

1. *Conditions of Lemma* 3.4 *hold.*
2. $m \leq x_i + I_{ik}(x_i) \leq M \, for \, m \leq x_i \leq M, i = 1, 2, ..., n, k = 1, 2, ...$
3. *The following inequalities are valid*

$$a_{jj}^L \geq \sum_{\substack{i=1 \\ i \neq j}}^{n} a_{ji}^M, \quad t \neq t_k, \quad k = 1, 2, ...$$

Then the solution $x^*(t)$ *of system* (1.1) *is uniformly stable.*

Proof. For $t \geq 0$, define a Lyapunov functional

$V(t, x(t), x^*(t)) =$

$$\sum_{i=1}^{n}\left[|\ln\frac{x_i(t)}{x_i^*(t)}| + \sum_{\substack{j=1\\ j\neq i}}^{n}\int_{t-\tau}^{t} a_{ij}(u+\tau)\sup_{s\in[u,t]}|x_j(s)-x_j^*(s)|du\right]. \quad (4.1)$$

Obviously,

$$V(t, x(t), x^*(t)) \geq \sum_{i=1}^{n}|\ln\frac{x_i(t)}{x_i^*(t)}|, \quad t \geq 0. \quad (4.2)$$

By the Mean Value Theorem and (3.5), it follows that for any closed interval contained in $[0, t_1] \cup (t_k, t_{k+1}]$, $k = 1, 2, ...$ and for all $i = 1, 2, ...$

$$\frac{1}{M}|x_i(t) - x_i^*(t)| \leq |\ln x_i(t) - \ln x_i^*(t)| \leq \frac{1}{m}|x_i(t) - x_i^*(t)|. \quad (4.3)$$

From (4.1) and (4.3), we obtain

$V(0^+, x(0^+), x^*(0^+)) =$

$$\sum_{i=1}^{n}\left[|\ln x_i(0^+) - \ln x_i^*(0^+)| + \sum_{\substack{j=1\\ j\neq i}}^{n}\int_{-\tau}^{0} a_{ij}(u+\tau)\sup_{s\in[u,0]}|x_j(s)-x_j^*(s)|du\right]$$

$$\leq \frac{1}{m}|\varphi_i(0) - \phi_i(0)| + \lambda||\varphi - \phi||_\tau \leq \left(\frac{1}{m} + \lambda\right)||\varphi - \phi||_\tau, \quad (4.4)$$

where $\lambda = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^M$.

For $t > 0$ and $t = t_k$, $k = 1, 2, ...$, we have
$V(t_k^+, x(t_k^+), x^*(t_k^+)) =$

$$\sum_{i=1}^{n}\left[|\ln\frac{x_i(t_k^+)}{x_i^*(t_k^+)}| + \sum_{\substack{j=1\\ j\neq i}}^{n}\int_{t_k^+-\tau}^{t_k^+} a_{ij}(u+\tau)\sup_{s\in[u,t]}|x_j(s)-x_j^*(s)|du\right]$$

$$= \sum_{i=1}^{n} \left[|\ln \frac{x_i(t_k)+I_{ik}(x_i(t_k))}{x_i^*(t_k)+I_{ik}(x_i^*(t_k))}| + \sum_{\substack{j=1 \\ j\neq i}}^{n} \int_{t_k-\tau}^{t_k} a_{ij}(u+\tau) \sup_{s\in[u,t]} |x_j(s)-x_j^*(s)|du \right]$$

$$\leq \sum_{i=1}^{n} \left[|\ln \frac{M}{m}| + \sum_{\substack{j=1 \\ j\neq i}}^{n} \int_{t_k-\tau}^{t_k} a_{ij}(u+\tau) \sup_{s\in[u,t]} |x_j(s) - x_j^*(s)|du \right]$$

$$\leq \sum_{i=1}^{n} \left[|\ln \frac{x_i(t_k)}{x_i^*(t_k)}| + \sum_{\substack{j=1 \\ j\neq i}}^{n} \int_{t_k-\tau}^{t_k} a_{ij}(u+\tau) \sup_{s\in[u,t]} |x_j(s) - x_j^*(s)|du \right]$$

$$= V(t_k, x(t_k), x^*(t_k)). \tag{4.5}$$

Consider the upper right-hand derivative $D^+_{(1.1)}V(t, x(t), x^*(t))$ of $V(t, x(t), x^*(t))$ with respect to system (1.1). For $t \geq 0$ and $t \neq t_k$, $k = 1, 2, ...$, we derive the estimate

$$D^+_{(1.1)}V(t, x(t), x^*(t))$$

$$\leq \sum_{i=1}^{n} \left[-a_{ii}(t)|x_i(t) - x_i^*(t)| + \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}(t+\tau)|x_j(t) - x_j^*(t)| \right]$$

$$= \sum_{j=1}^{n} \left[-a_{jj}(t)|x_j(t) - x_j^*(t)| + \sum_{\substack{i=1 \\ i\neq j}}^{n} a_{ij}(t+\tau)|x_j(t) - x_j^*(t)| \right]$$

$$\leq \sum_{j=1}^{n} \left[-a_{jj}^L|x_j(t) - x_j^*(t)| + \sum_{\substack{i=1 \\ i\neq j}}^{n} \left(a_{ij}^M \right) |x_j(t) - x_j^*(t)| \right].$$

¿From the last inequality and condition 3 of Theorem 4.1, we obtain

$$D^+_{(1.1)}V(t, x(t), x^*(t)) \leq 0, \tag{4.6}$$

$t \geq 0$ and $t \neq t_k$, $k = 1, 2, ...$

Given $0 < \varepsilon < M$, choose $\delta = \frac{\varepsilon m}{2M(1+\lambda m)}$. Then, from (4.2), (4.4), (4.5) and (4.6), we obtain

$$||x(t) - x^*(t)|| \leq MV(t, x(t), x^*(t)) \leq MV(0^+, x(0^+), x^*(0^+))$$

$$\leq M\left(\frac{1}{m}+\lambda\right)||\varphi-\phi||_\tau \leq \varepsilon,$$

$t \geq 0$. This shows that the solution $x^*(t)$ of system (1.1) is uniformly stable.

Theorem 4.2. *In addition to the assumptions of Theorem* 4.1, *suppose there exists a nonnegative constant* μ *such that*

$$a_{jj}^L \geq \mu + \left(\sum_{\substack{i=1\\ i\neq j}}^n a_{ji}^M\right), \quad t \neq t_k, \quad k=1,2,.... \tag{4.7}$$

Then the solution $x^*(t)$ *of system* (1.1) *is uniformly asymptotically stable.*

Proof. We consider again the Lyapunov functional (4.1). From (4.3) and (4.7), we obtain

$$D^+_{(1.1)}V(t,x(t),x^*(t)) \leq -\frac{\mu}{m}\sum_{i=1}^n |x_i(t)-x_i^*(t)| \leq -\mu V(t,x(t),x^*(t)),$$

$t \geq 0$ and $t \neq t_k$, $k=1,2,....$

¿From the last inequality and (4.5), we have

$$V(t,x(t),x^*(t)) \leq V(0^+,x(0^+),x^*(0^+))exp\{-\mu t\} \tag{4.8}$$

for all $t \geq t_0$. Then, from (4.2), (4.8) and (4.3) we deduce the inequality

$$\sum_{i=1}^n |x_i(t)-x_i^*(t)| \leq M\left(\frac{1}{m}+\lambda\right)||\varphi-\phi||_\tau e^{-\mu t},$$

$t \geq 0$.

This shows that the solution $x^*(t)$ of system (1.1) is uniformly asymptotically stable. The proof of Theorem 4.2 is complete.

5 Applications

In order to illustrate some feature of our main results, in the following we will apply Theorem 4.2 to a two-species system. The results obtained can be applied in the investigation of the stability of any solution which is of interest.

One of the solutions which is an object of investigations for the systems of type (1.1) is the positive *periodic solution.* To consider periodic environmental factors, it is reasonable to study the Lotka-Volterra systems with periodic coefficients. The assumption of periodicity of the parameters r_i, a_{ij} is a way of incorporating environmental periodicity (e.g. seasonal effects of weather condition, food supplies, temperature, etc).

The existence and stability of *equilibrium* states of some special cases of (1.1) without impulses has been studied extensively in the literature. In this case we do not need the assumptions of periodicity on the parameters. Let $\tau > 0$ be a constant.

Example 5.1. For the system

$$\begin{cases} \dot{x}(t) = x(t)\left[6 - 15x(t) - \sup\limits_{s\in[t-\tau,t]} y(s)\right], \\ \dot{y}(t) = y(t)\left[17 - 3\sup\limits_{s\in[t-\tau,t]} x(s) - 16y(t)\right], \end{cases} \tag{5.1}$$

one can show that the point $(x^*, y^*) = (\frac{1}{3}, 1)$ is an equilibrium which is uniformly asymptotically stable [7].

Now, we consider the impulsive Lotka-Volterra system

$$\begin{cases} \dot{x}(t) = x(t)\left[6 - 15x(t) - \sup\limits_{s\in[t-\tau,t]} y(s)\right], \ t \neq t_k, \\ \\ \dot{y}(t) = y(t)\left[17 - 3\sup\limits_{s\in[t-\tau,t]} x(s) - 16y(t)\right], \ t \neq t_k, \\ \\ \Delta x(t_k) = -\frac{2}{5}\Big(x(t_k) - \frac{1}{3}\Big), \ k = 1, 2, ..., \\ \\ \Delta y(t_k) = -\frac{3}{5}\Big(y(t_k) - 1\Big), \ k = 1, 2, ..., \end{cases} \tag{5.2}$$

where $0 < t_1 < t_2 < ...$ and $\lim\limits_{k\to\infty} t_k = \infty$.

For the system (5.2), the point $(x^*, y^*) = (\frac{1}{3}, 1)$ is an equilibrium and all conditions of Theorem 4.2 are satisfied. In fact, for $\mu \leq 1$, $m = \frac{1}{3}$ and $M = 1$, we have

$$\frac{1}{3} \leq x(t_k) + I_{1k}(x(t_k)) = \frac{9x(t_k) + 2}{15} \leq 1,$$
$$\frac{1}{3} \leq y(t_k) + I_{2k}(y(t_k)) = \frac{2y(t_k) + 3}{5} \leq 1$$

for $\frac{1}{3} \leq x(t_k) \leq 1$, $\frac{1}{3} \leq y(t_k) \leq 1$, $k = 1, 2, ...$

Therefore, the equilibrium $(x^*, y^*) = (\frac{1}{3}, 1)$ is uniformly asymptotically stable.

If, in the system (5.2), we consider the impulsive perturbations of the form:

$$\begin{cases} \Delta x(t_k) = -3\Big(x(t_k) - \frac{1}{3}\Big), \ k = 1, 2, ..., \\ \Delta y(t_k) = -\frac{3}{5}\Big(y(t_k) - 1\Big), \ k = 1, 2, ..., \end{cases}$$

then the point $(x^*, y^*) = (\frac{1}{3}, 1)$ is again an equilibrium, but there is nothing we can say about its uniform asymptotic stability, because for $\frac{1}{3} \leq x(t_k) \leq 1$, we have $-1 \leq x(t_k) + I_{1k}(x(t_k)) \leq \dfrac{1}{3}$, $k = 1, 2, ...$

The example shows that by means of appropriate impulsive perturbations we can control the system's population dynamics. We can see that impulses are used to keep the stability properties of the system. On the other hand, a well-behaved system may lose its (asymptotic) stability due to uncontrolled impulsive inputs. Theorem 4.2 provides a set of sufficient conditions under which the asymptotic stability properties of a Lotka-Volterra system can be preserved under impulsive perturbations.

Example 5.2. The system

$$\begin{cases} \dot{x}(t) = x(t)\left[5 - 15x(t) - 2\sup\limits_{s\in[t-\tau,t]} y(s)\right], \\ \\ \dot{y}(t) = y(t)\left[\dfrac{201}{10} - \dfrac{1}{2}\sup\limits_{s\in[t-\tau,t]} x(s) - 20y(t)\right], \end{cases} \tag{5.3}$$

has an uniformly asymptotically stable equilibrium point $(x^*, y^*) = (\frac{1}{3}, 0)$ which implies that the second species will be driven to extinction.

However, for the impulsive Lotka-Volterra system

$$\begin{cases} \dot{x}(t) = x(t)\left[5 - 15x(t) - 2\sup\limits_{s\in[t-\tau,t]} y(s)\right], \ t \neq t_k, \\ \\ \dot{y}(t) = y(t)\left[\dfrac{201}{10} - \dfrac{1}{2}\sup\limits_{s\in[t-\tau,t]} x(s) - 20y(t)\right], \ t \neq t_k, \\ \\ \Delta x(t_k) = -\dfrac{1}{2}\Big(x(t_k) - \dfrac{1}{5}\Big), \ k = 1, 2, ..., \\ \\ \Delta y(t_k) = -\dfrac{1}{3}\Big(y(t_k) - 1\Big), \ k = 1, 2, ..., \end{cases}$$

where $0 < t_1 < t_2 < ...$ and $\lim\limits_{k\to\infty} t_k = \infty$, the point $(x^*, y^*) = (\frac{1}{5}, 1)$ is an equilibrium which is uniformly asymptotically stable. In fact, all conditions of Theorem 4.2 are satisfied for $\mu \le \frac{1}{2}$, $m = \frac{1}{5}$, $M = 1$ and

$$\frac{1}{5} \le x(t_k) + I_{1k}(x(t_k)) = \frac{5x(t_k)+1}{10} \le 1,$$

$$\frac{1}{5} \le y(t_k) + I_{2k}(y(t_k)) = \frac{2y(t_k)+1}{3} \le 1$$

for $\frac{1}{5} \le x(t_k) \le 1$, $\frac{1}{5} \le y(t_k) \le 1$, $k = 1, 2, \ldots.$

This shows that the impulsive perturbations can prevent the population from going extinct.

Acknowledgements

The authors are thankful to anonymous reviewer for the thoughtful suggestions that lead to an improved exposition of this paper.

References

[1] S. Ahmad and I.M. Stamova, Lotka-Volterra and Related Systems: Recent Developments in Population Dynamics, Walter de Gruyter, Berlin, 2013.

[2] S. Ahmad and G.T. Stamov, Almost periodic solutions of N-dimensional impulsive competitive systems, *Nonlinear Anal. RWA* **10**, (2009), 1846–1853.

[3] J. Angelova and A. Dishliev, Optimization problems for one-impulsive models from population dynamics, Nonlinear Anal. **39** (2000), 483–497.

[4] D.D. Bainov and P.S. Simeonov, Impulsive Differential Equations: Periodic Solutions and Applications, Essex, Longman, 1993.

[5] M. Benchohra, J. Henderson, S.K. Ntouyas and A. Ouahabi, Higher order impulsive functional differential equations with variable times, *Dynam. Systems Appl.* **12** (2003), 383–392.

[6] K. Dishlieva and A. Dishlieva, Stability of impulsive differential equations with random impulsive moments, Int. J. Sci. Techn. Manag. **4** (2015), 162–172.

[7] A. D'onofrio, Stability properties of pulse vaccination strategy in SEIR epidemic model, *Math. Biosci.* **179** (2002), 57–72.

[8] J.W. Dou, L.S. Chen and K.T. Li, A monotone-iterative method for finding periodic solutions of an impulsive competition system on tumor-normal cell interaction, *Discrete Contin. Dyn. Syst. Ser. B* **4** (2004), 555–562.

[9] K. Gopalsamy, Stability and Oscillation in Delay Differential Equations of Population Dynamics, Mathematics and its Applications, vol. 74. Dordrecht: Kluwer Academic; 1992.

[10] V. Lakshmikantham, D D. Bainov and P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, New Jersey, London, 1989.

[11] A. Lakmeche and O. Arino, Nonlinear mathematical model of pulsed therapy of heterogeneous tumors, *Nonlinear Anal.* **2** (2001), 455–465.

[12] M. Li, Y. Duan, W. Zhang and M. Wang, The existence of positive periodic solutions of a class of Lotka-Volterra type impulsive systems with infinitely distributed delay, *Comput. Math. Appl.* **49** (2005), 1037–1044.

[13] B. Liu, L.S. Chen and Y.J. Zhang, The effects of impulsive toxicant input on a population in a polluted environment, *J. Biol. Syst.* **11** (2003), 265–274.

[14] X.Z. Liu and K. Rohlf, Impulsive control of Lotka-Volterra models, *IMA J. Math. Control Infor.* **15** (1998), 269-284.

[15] Z. Liu, J. Wu and R. Tang, Permanence and extinction of an impulsive delay competitive Lotka-Volterra model with periodic coefficients, *IMA J. Appl. Math.* **74** (2009) 559–573.

[16] N.G. Medhin and M. Sambandhan, On the control of impulsive SIR models, *Commun. Appl. Anal.* **17** (2013), 367–378.

[17] G.T. Stamov, Almost Periodic Solutions of Impulsive Differential Equations, Springer, Berlin, 2012.

[18] I.M. Stamova, Lyapunov-Razumikhin method for impulsive differential equations with 'supremum', IMA J. Appl. Math. 76 (2011), 573–581.

[19] I.M. Stamova and G.T. Stamov, Applied Impulsive Mathematical Models, Springer, CMS Books in Mathematics, Cham, 2016.

[20] S.L. Sun and L.S. Chen, Dynamic behaviors of Monod type chemostat model with impulsive perturbation on the nutrient concentration, *J. Math. Chem.* **42** (2007), 837–848.

[21] Y. Takeuchi, Global Dynamical Properties of Lotka-Volterra Systems, World Scientific, Singapore, 1996.