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The new trasmuted C.D.F. based on Gompertz function

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This article is dedicated to 75th anniversary of Professor, Sc. D. Svetoslav Markov

Abstract.

In this paper we find application of some new cumulative distribution functions transformations to construct a family of sigmoidal functions based on the Gompertz logistic function.

We prove estimates for the Hausdorff approximation of the shifted Heaviside step function by means of these families.

Numerical examples, illustrating our results are given.

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1 Introduction

In literature, several transformations exists to obtain a new cumulative distribution function (cdf) using other(s) well-known cdf(s) [1]–[7].

Many researches have used the quadratic rank transmuted map (QRTM) to develop new life time distribution.

Definition. Another popular transformation by using a (cdf) $F(t)$ is [5]:

$$G(t) = \frac{1}{e-1}\left(e^{F(t)} - 1\right). \tag{1}$$

The transformation (1) has great applications in data analysis.

Definition. Another popular transformation by using a (cdf) $F(t)$ is [7]:

$$G1(t) = e^{1-\frac{1}{F(t)}}. \tag{2}$$

Kumar et al. [8] proposed the cdf distribution by the use of any two cdf $F_1(t)$ and $F_2(t)$ of baseline continuous distribution(s) with common spectrum, by the transformation:

Definition. [8]

$$G2(t) = \frac{F_1(t) + F_2(t)}{1 + F_1(t)}. \tag{3}$$

If $F_1(t) = F_2(t) = F(t)$, then (3) reduces to the following form

$$G2(t) = \frac{2F(t)}{1 + F(t)}. \tag{4}$$

The transformation (3) has great applications in life time analysis.

Definition. Define the logistic (Verhulst) function f on $\mathbb{R}$ as

$$f(t) = \frac{1}{1 + e^{-kt}}. \tag{5}$$

The logistic function belongs to the important class of smooth sigmoidal functions arising from population and cell growth models.

Since then the logistic function finds applications in many scientific fields, including biology, population dynamics, chemistry, demography, economics, geoscience, mathematical psychology, probability, financial mathematics, statistics, insurance mathematics, nucleation theory to name a few [9]–[18], [41]–[47].

Definition. The (interval) step function is:

$$h_{t_0}(t) = \begin{cases} 0, & \text{if} \quad t < t_0, \\ [0,1], & \text{if} \quad t = t_0, \\ 1, & \text{if} \quad t > t_0, \end{cases}$$

usually known as *shifted Heaviside step function.*

Definition. [19], [20] The Hausdorff distance (the H–distance) $\rho(f,g)$ between two interval functions f, g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,

$$\rho(f,g) = \max\{ \sup_{A \in F(f)} \inf_{B \in F(g)} ||A - B||, \sup_{B \in F(g)} \inf_{A \in F(f)} ||A - B||\}, \quad (6)$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t,x)|| = \max\{|t|,|x|\}$; hence the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in $\mathbb{R}^2$ is $||A - B|| = \max(|t_A - t_B|, |x_A - x_B|)$.

Definition. The Gompertz function $\sigma_{\alpha,\beta}(t)$ is defined for α, $\beta > 0$ by [31]–[33]:

$$\sigma_{\alpha,\beta}(t) = ae^{-\alpha e^{-\beta t}}, \quad (7)$$

where a is the upper asymptote when time approaches $+\infty$.

Gompertz functions, are introduced by Benjamin Gompertz for the study of his demographic model, which represents a refinement of the Malthus model.

The functions find applications in modeling tumor growth and in population aging description.

In biology, the Gompertz curve or Gompertz function is commonly used to model growth process where the period of increasing growth is shorter than the period in which growth decreases.

For other results, see [34]–[40].

Let $F(t) = f(t)$. Then the following theorems are valid

Theorem A. [**30**] *The one–sided H-distance $d(k)$ between the function h_{t_0} and the function G can be expressed in terms of the rate parameter k for any real $k \geq 2$ as follows:*

$$d_l = \frac{1}{2.5(1 + 0.254884k)} < d < \frac{\ln(2.5(1 + 0.254884k))}{2.5(1 + 0.254884k)} = d_r. \quad (8)$$

Theorem B. [**30**] *The one–sided H-distance $d_1(k)$ between the function h_{t_0} and the function $G1$ can be expressed in terms of the rate parameter k for any real $k \geq 2$ as follows:*

$$d_{l_1} = \frac{1}{2.5(1 + 0.346574k)} < d_1 < \frac{\ln(2.5(1 + 0.346574k))}{2.5(1 + 0.346574k)} = d_{r_1}. \quad (9)$$

Theorem C. [**41**] *The H-distance $d_2(k)$ between the function h_{t_0} and the function $G2$ can be expressed in terms of the rate parameter k for any real $k \geq 2$ as follows:*

$$d_{l_2} = \frac{1}{2.5(1 + 0.25k)} < d_2 < \frac{\ln(2.5(1 + 0.25k))}{2.5(1 + 0.25k)} = d_{r_2}. \quad (10)$$

2 Main Results

In this Section we discuss several computational, modelling and approximation issues related to the class of cdf transformation (2)–(4) to construct a family of sigmoidal functions based on the Gompertz logistic function.

2.1 Type III.

Let us consider the following sigmoid

$$G^*(t) = \frac{2e^{-e^{-\beta t}}}{1 + e^{-e^{-\beta t}}} \tag{11}$$

with

$$G^*(t_0) = \frac{1}{2}, \quad t_0 = -\frac{1}{\beta}\ln(\ln 3) \tag{12}$$

based on (4) with the Gompertz logistic function $F(t) = \sigma_{1,\beta}(t) = e^{-e^{-\beta t}}$.

The H-distance $d_3 = \rho(h_{t_0}, G^*)$ between the shifted Heaviside step function h_{t_0} and the sigmoidal function G^* satisfies the relation:

$$G^*(t_0 + d_3) = \frac{2e^{-e^{-\beta(t_0+d_3)}}}{1 + e^{-e^{-\beta(t_0+d_3)}}} = 1 - d_3. \tag{13}$$

The following theorem gives upper and lower bounds for $d_3 = d_3(\beta)$

Theorem 1 *The one–sided H-distance $d_3(\beta)$ between the function h_{t_0} and the function G^* can be expressed in terms of the parameter β for any real $\beta \geq 2.41793$ as follows:*

$$d_{l_3} = \frac{1}{2.5(1 + 0.41198\beta)} < d_3 < \frac{\ln(2.5(1 + 0.41198\beta))}{2.5(1 + 0.41198\beta)} = d_{r_3}. \tag{14}$$

Proof. We define the functions

$$F_3(d_3) = \frac{2e^{-e^{-\beta(t_0+d_3)}}}{1 + e^{-e^{-\beta(t_0+d_3)}}} - 1 + d_3 \tag{15}$$

$$G_3(d_3) = -\frac{1}{2} + (1 + 0.41198\beta)d_3. \tag{16}$$

From Taylor expansion

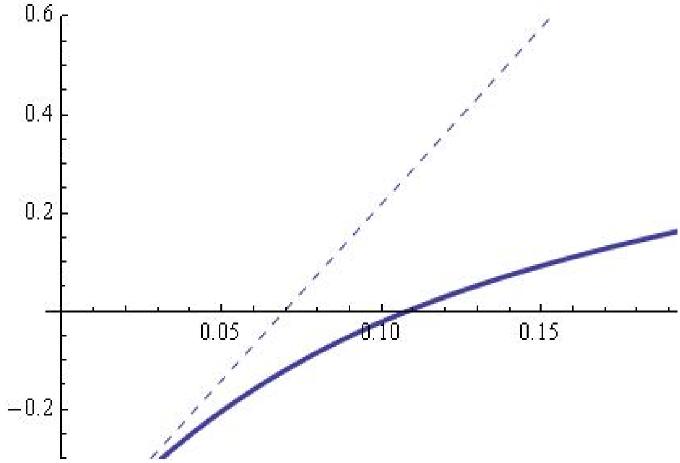

Figure 1: The functions $F_3(d)$ and $G_3(d)$ for $\beta = 15$.

$$\frac{2e^{-e^{-\beta(t_0+d_3)}}}{1+e^{-e^{-\beta(t_0+d_3)}}} - 1 + d_3 - (-\frac{1}{2} + (1 + 0.41198\beta)d_3) = O(d_3^2)$$

we see that the function $G_3(d_3)$ approximates $F_3(d_3)$ with $d_3 \to 0$ as $O(d_3^2)$ (cf. Fig. 1).

In addition $G_3'(d_3) > 0$ and for $\beta \geq 2.41793$

$$G_3(d_{l_3}) < 0; \quad G_3(d_{r_3}) > 0.$$

This completes the proof of the inequalities (14).

The generated sigmoidal functions $G^*(t)$ for $\beta = 7, 15$ and 18 are visualized on Fig. 2–Fig. 4.

Some computational examples using relations (14) are presented in Table 1.

The third column of Table 1 contains the value of d_3 for prescribed values of β computed by solving the nonlinear equation (13).

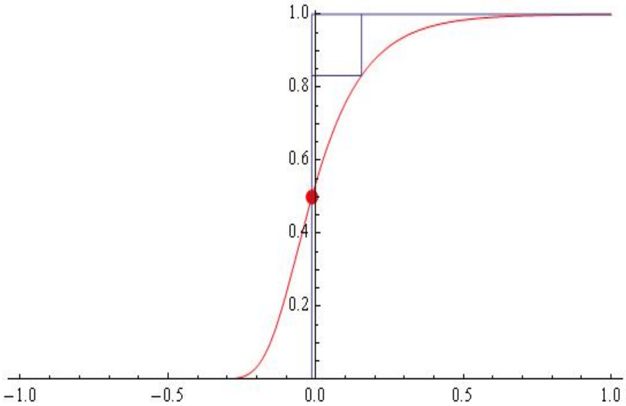

Figure 2: The H-distance $d_3(\beta)$ between the functions h_{t_0} and G^* for $\beta = 7$ is $d_3 = 0.167936$; $d_{l_3} = 0.10299$; $d_{r_3} = 0.234109$; $t_0 = -0.0134354$.

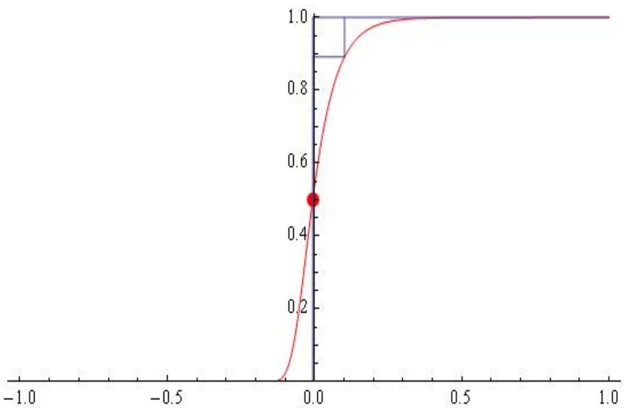

Figure 3: The H-distance $d_3(\beta)$ between the functions h_{t_0} and G^* for $\beta = 15$ is $d_3 = 0.108108$; $d_{l_3} = 0.0557126$; $d_{r_3} = 0.160873$; $t_0 = -0.00626986$.

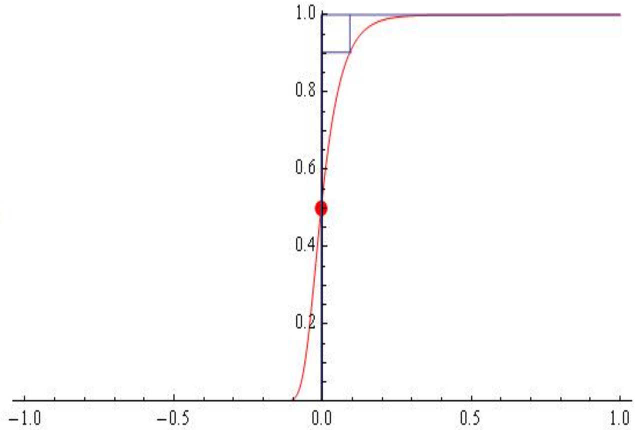

Figure 4: The H-distance $d_3(\beta)$ between the functions h_{t_0} and G^* for $\beta = 18$ is $d_3 = 0.0964647$; $d_{l_3} = 0.0475306$; $d_{r_3} = 0.144796$; $t_0 = -0.00522488$.

2.2 Type II.

Let us consider the following sigmoid

$$G1^*(t) = e^{1-\frac{1}{e^{-e^{-\beta t}}}} \tag{17}$$

with

$$G1^*(t_0) = \frac{1}{2}, \quad t_0 = -\frac{1}{\beta}\ln(\ln(1+\ln 2)) \tag{18}$$

based on (2) with the Gompertz function $F(t) = \sigma_{1,\beta}(t) = e^{-e^{-\beta t}}$.

The H-distance $d_4 = \rho(h_{t_0}, G1^*)$ between the shifted Heaviside step function h_{t_0} and the sigmoidal function $G1^*$ satisfies the relation:

$$G1^*(t_0 + d_4) = e^{1-\frac{1}{e^{-e^{-\beta(t_0+d_4)}}}} = 1 - d_4. \tag{19}$$

```
Manipulate[Dynamic@Show[Plot[G[t], {t, -1, 1}, LabelStyle →
      Directive[Green, Bold],
     PlotLabel → 2 * Exp[-Exp[-β * t]] / (1 + Exp[-Exp[-β * t]])],
   PlotRange → {Automatic, {0, 1}}, AxesOrigin → {0, 0}],
 {{β, 1}, 1, 100, Appearance → "Open"},
 {{t0, 0}, -0.007, 10, Appearance → "Open"},
 Initialization :→ (G[t_] := 2 * Exp[-Exp[-β * t]] / (1 + Exp[-Exp[-β * t]]))]
```

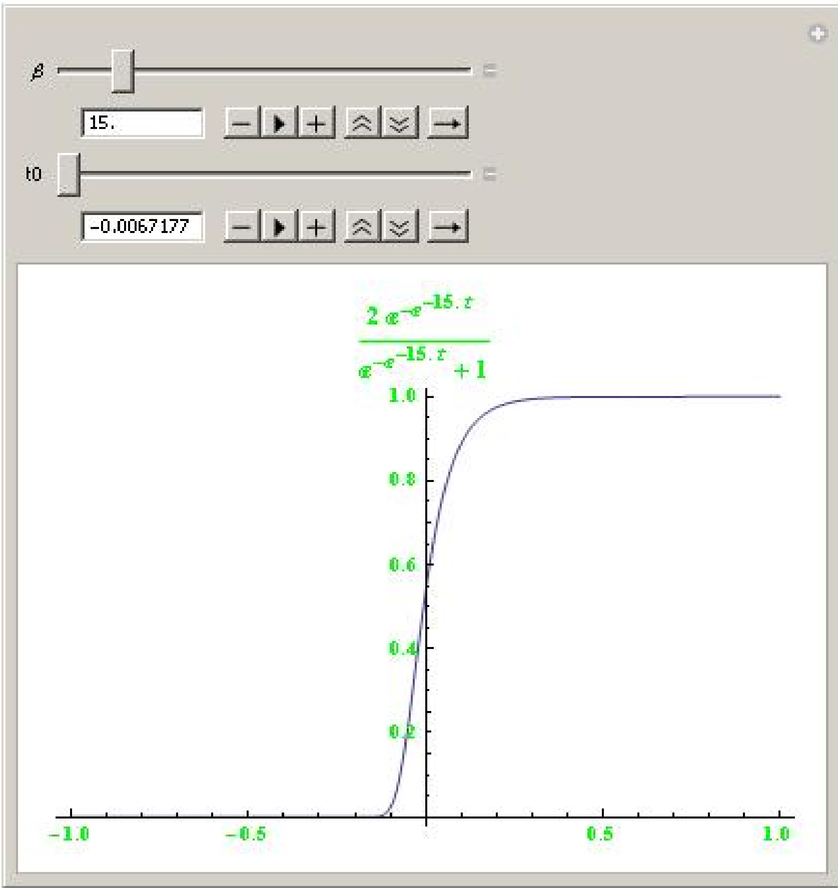

Figure 5: Software tools in CAS Mathematica.

β	d_l	d_3 computed by (13)	d_r
7	0.10299	0.167936	0.234109
15	0.0557126	0.108108	0.160873
200	0.00479639	0.0172917	0.0256122
300	0.00321043	0.0125866	0.0184322
500	0.00193246	0.00836835	0.0120759
600	0.00161168	0.00721971	0.0103639
700	0.00138224	0.00636772	0.00910072
1000	0.00096857	0.00475042	0.00672157

Table 1: Bounds for $d_3(\beta)$ computed by (13) and (14) for various β

Based on the methodology proposed in the present note, the reader may formulate the corresponding modeling and approximation problems on his/her own.

The following theorem gives upper and lower bounds for $d_4 = d_4(\beta)$

Theorem 2 *The one–sided H-distance $d_4(\beta)$ between the function h_{t_0} and the function $G1^*$ can be expressed in terms of the parameter β for any real $\beta \geq 0.888606$ as follows:*

$$d_{l_4} = \frac{1}{2.5(1 + 0.445796\beta)} < d_4 < \frac{\ln(2.5(1 + 0.445796\beta))}{2.5(1 + 0.445796\beta)} = d_{r_4}. \quad (20)$$

Proof. We define the functions

$$F_4(d_4) = e^{1 - \frac{1}{e^{-e^{-\beta(t_0+d_4)}}}} - 1 + d_4 \quad (21)$$

$$G_4(d_4) = -\frac{1}{2} + (1 + 0.445796\beta)d_4. \quad (22)$$

From Taylor expansion

$$e^{1 - \frac{1}{e^{-e^{-\beta(t_0+d_4)}}}} - 1 + d_4 - (-\frac{1}{2} + (1 + 0.445796\beta)d_4) = O(d_4^2)$$

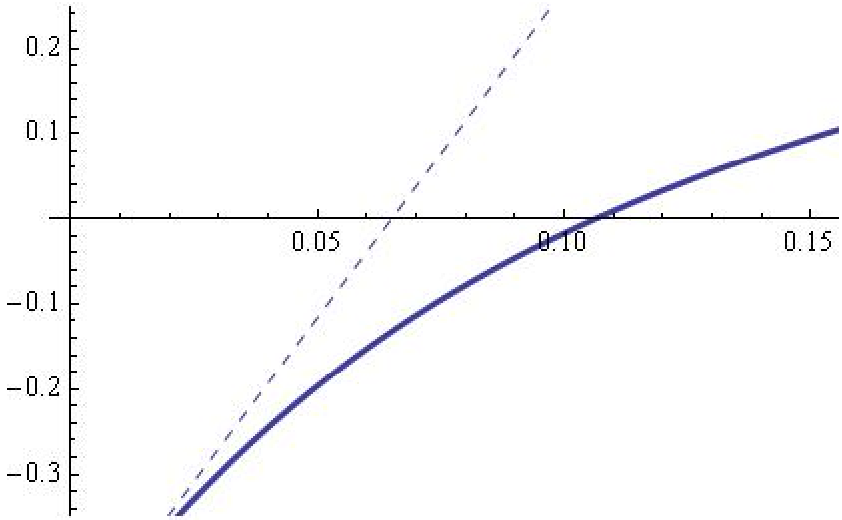

Figure 6: The functions $F_3(d)$ and $G_3(d)$ for $\beta = 15$.

we see that the function $G_4(d_4)$ approximates $F_4(d_4)$ with $d_4 \to 0$ as $O(d_4^2)$ (cf. Fig. 6).

In addition $G_4'(d_4) > 0$ and for $\beta \geq 0.888606$

$$G_4(d_{l_4}) < 0; \quad G_4(d_{r_4}) > 0.$$

This completes the proof of the inequalities (20).

The generated sigmoidal functions $G1^*(t)$ for $\beta = 7, 15$ are visualized on Fig. 7–Fig. 9.

From the graphics it can be seen that the "saturation" is faster.

2.3 Type I.

Let us consider the following sigmoid

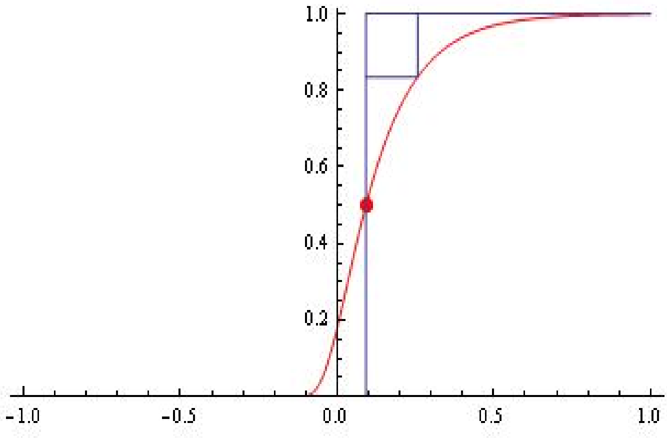

Figure 7: The H-distance $d_4(\beta)$ between the functions h_{t_0} and $G1^*$ for $\beta = 7$ is $d_4 = 0.165054$; $d_{l_4} = 0.0970739$; $d_{r_4} = 0.226404$; $t_0 = 0.0916193$.

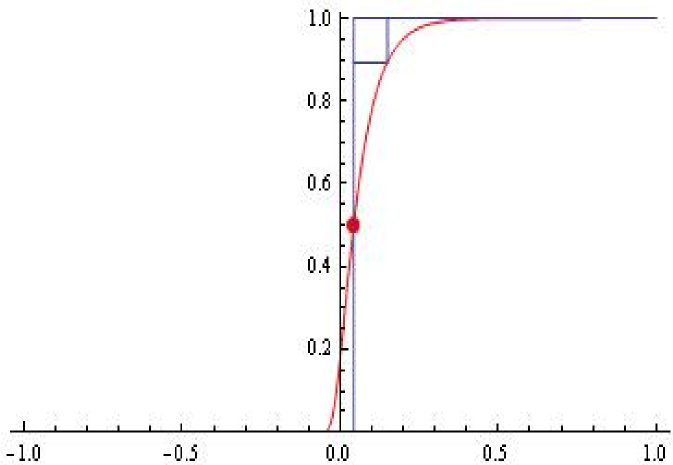

Figure 8: The H-distance $d_4(\beta)$ between the functions h_{t_0} and $G1^*$ for $\beta = 15$ is $d_4 = 0.106452$; $d_{l_4} = 0.0520363$; $d_{r_4} = 0.15381$; $t_0 = 0.00427557$.

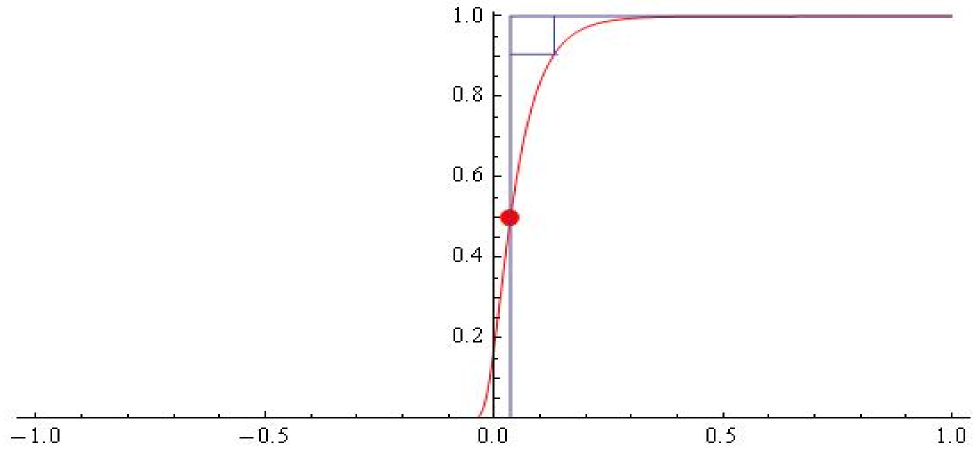

Figure 9: The H-distance $d_4(\beta)$ between the functions h_{t_0} and $G1^*$ for $\beta = 18$ is $d_4 = 0.095035$; $d_{l_4} = 0.0443246$; $d_{r_4} = 0.138125$.

$$G3^*(t) = \frac{1}{e-1}\left(e^{e^{-e^{-\beta t}}} - 1\right) \tag{23}$$

with

$$G3^*(t_0) = \frac{1}{2}, \quad t_0 = -\frac{1}{\beta}\ln(-\ln(\ln\frac{1+e}{2})) \tag{24}$$

based on (1) with the Gompertz function $F(t) = \sigma_{1,\beta}(t) = e^{-e^{-\beta t}}$.

The H-distance $d_5 = \rho(h_{t_0}, G3^*)$ between the shifted Heaviside step function h_{t_0} and the sigmoidal function $G3^*$ satisfies the relation:

$$G3^*(t_0 + d_5) = \frac{1}{e-1}\left(e^{e^{-e^{-\beta(t_0+d_5)}}} - 1\right) = 1 - d_5. \tag{25}$$

Based on the methodology proposed in the present note, the reader may formulate the corresponding modeling and approximation problems on his/her own.

The following theorem gives upper and lower bounds for $d_5 = d_5(\beta)$

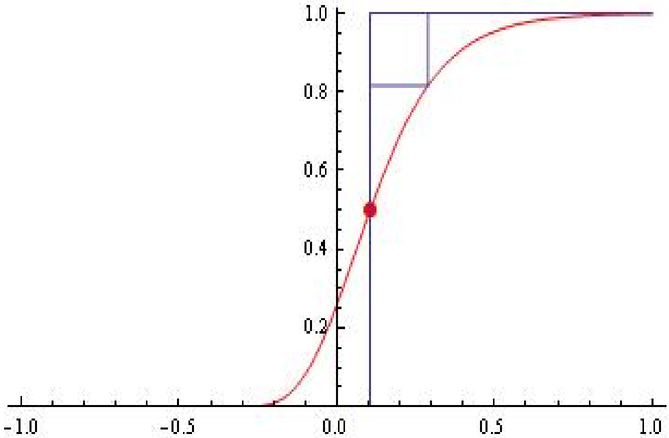

Figure 10: The H-distance $d_5(\beta)$ between the functions h_{t_0} and $G3^*$ for $\beta = 7$ is $d_5 = 0.183892$; $d_{l_5} = 0.123293$; $d_{r_5} = 0.258076$; $t_0 = 0.105494$.

Theorem 3 *The one–sided H-distance $d_5(\beta)$ between the function h_{t_0} and the function $G3^*$ can be expressed in terms of the parameter β for any real $\beta \geq 1.23556$ as follows:*

$$d_{l_5} = \frac{1}{2.5(1 + 0.320614\beta)} < d_5 < \frac{\ln(2.5(1 + 0.320614\beta))}{2.5(1 + 0.320614\beta)} = d_{r_5}. \quad (26)$$

The proof follows the ideas given in this paper and will be omitted.

3 Conclusions

To achieve our goal, we obtain new estimates for the H-distance between a shifted Heaviside step function and its best approximating family of transmuted cumulative distribution functions $G^*(t)$ and $G1^*(t)$ based on the Gompertz function.

The result has application in population dynamics and neural networks.

Numerical examples, illustrating our results are given.

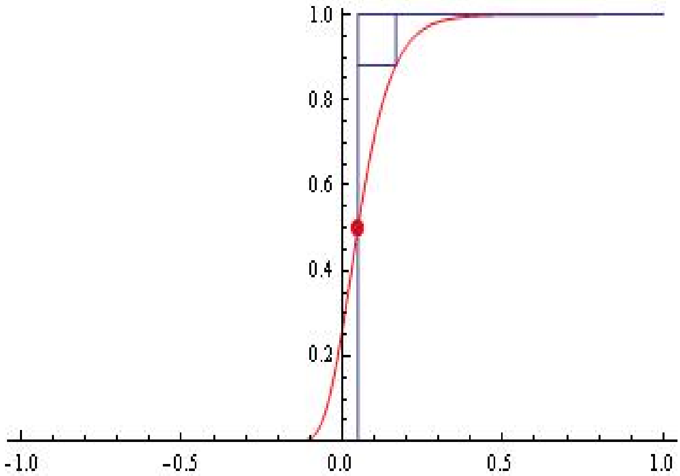

Figure 11: The H-distance $d_5(\beta)$ between the functions h_{t_0} and $G3^*$ for $\beta = 15$ is $d_5 = 0.11836$; $d_{l_5} = 0.0688562$; $d_{r_5} = 0.184241$; $t_0 = 0.0492304$.

For other results, see [21]–[29].

We propose a software module within the programming environment *CAS Mathematica* for the analysis of the considered families of transmuted cumulative distribution functions.

The module offers the following possibilities:

- generation of the function $G^*(t)$ under user defined values of the β and t_0;

- calculation of the H-distance between the Heaviside function h_{t_0} and the sigmoidal function $G^*(t)$;

- software tools for animation and visualization.

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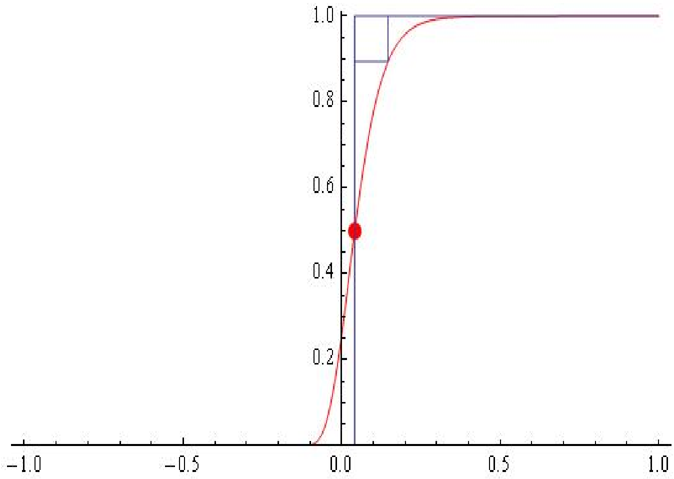

Figure 12: The H-distance $d_5(\beta)$ between the functions h_{t_0} and $G3^*$ for $\beta = 18$ is $d_5 = 0.105515$; $d_{l_5} = 0.059075$; $d_{r_5} = 0.16712$; $t_0 = 0.0410253$.

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