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On the Kumaraswamy–Dagum–Log–Logistic sigmoid functions with applications to population dynamics

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This article is dedicated to 75th anniversary of Professor, Sc. D. Svetoslav Markov

Abstract.

The Kumaraswamy–Dagum distribution is a flexible and simple model with applications to income and lifetime data.

We prove upper and lower estimates for the Hausdorff approximation of the shifted Heaviside function $\tilde{h}_{t_0}(t)$ by a class of Kumaraswamy–Dagum–Log–Logistic cumulative distribution function – (KD–CDF). Numerical examples, illustrating our results are given.

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1. Introduction. Dagum (1977) [3] motivates his model from the empirical observation that the income elasticity $\eta(F,t)$ of the cumulative distribution function (CDF) F of income is a decreasing and bounded function F.

The cumulative distribution function (cdf) of Dagum distribution is given by

$$G_D(t,\lambda,\beta,\delta) = \left(1-\lambda t^{-\delta}\right)^{-\beta}, \tag{1}$$

for $t \geq 0$, where λ is a scale parameter; δ and β are shape parameters.

The cumulative distribution function (cdf) of Kumuraswamy distribution is given by [10]

$$G(t) = 1-\left(1-t^{\psi}\right)^{\phi}, \quad t \in (0,1) \tag{2}$$

for $\psi > 0$ and $\phi > 0$.

This approach was further developed in a series of papers on generating systems for income distribution [4]–[7].

For other results, see [8], [9], [10] [11].

For an arbitrary (cdf) $F(t)$ with (PDF) $f(t) = \frac{dF(t)}{dt}$ the family of Kumaraswamy–G distribution with (cdf) $G_k(t)$ is given by

$$G_k(t) = 1-\left(1-F^{\psi}(t)\right)^{\phi}, \tag{3}$$

for $\psi > 0$ and $\phi > 0$.

By letting $F(t) = G_D(t)$, we obtain the Kumuraswamy–Dagum (KD) distribution, with (cdf)

$$G_{KD}(t) = 1-\left(1-G_D^{\psi}(t)\right)^{\phi}, \tag{4}$$

i.e.

$$G_{KD}(t) = 1 - \left(1 - \left(\left(1 + \lambda t^{-\delta}\right)^{-\beta}\right)^{\psi}\right)^{\phi}. \tag{5}$$

See [10] for further details.

When $\beta = 1$, we obtain Kumaraswamy–Dagum–Log–Logistic cumulative distribution function – (KD–CDF):

$$G_{KD}(t) = 1 - \left(1 - \left(\left(1 + \lambda t^{-\delta}\right)^{-1}\right)^{\psi}\right)^{\phi}. \tag{6}$$

In this paper we prove upper and lower estimates for the Hausdorff approximation of the shifted Heaviside function $\tilde{h}_{t_0}(t)$ by a class of Kumaraswamy–Dagum–Log–Logistic cumulative distribution function – (KD–CDF).

2. Preliminaries.

Definition 1. *The (basic) step function is:*

$$\tilde{h}_{t_0}(t) = \begin{cases} 0, & if \quad t < t_0, \\ [0,1], & if \quad t = t_0, \\ 1, & if \quad t > t_0, \end{cases} \tag{7}$$

usually known as shifted Heaviside function.

Definition 2. *[12], [13] The Hausdorff distance (the H–distance) [12] $\rho(f,g)$ between two interval functions f, g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$.*

More precisely,

$$\rho(f,g) = \max\{\sup_{A\in F(f)} \inf_{B\in F(g)} ||A-B||, \sup_{B\in F(g)} \inf_{A\in F(f)} ||A-B||\}, \tag{8}$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t,x)|| = \max\{|t|,|x|\}$; hence the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in $\mathbb{R}^2$ is $||A - B|| = max(|t_A - t_B|, |x_A - x_B|)$.

Let us point out that the Hausdorff distance is a natural measuring criteria for the approximation of bounded discontinuous functions [1].

3. Main Results.

Let us consider the following five parametric sigmoid function

$$F^*(t) = 1 - \left(1 - \left(\left(1 + \lambda t^{-\delta}\right)^{-\beta}\right)^{\psi}\right)^{\phi} \tag{9}$$

with

$$F^*(t_0) = \frac{1}{2}, \quad t_0 = \left(\frac{1}{\lambda}\left(\left(1 - 0.5^{\frac{1}{\phi}}\right)^{-\frac{1}{\beta\psi}} - 1\right)\right)^{-\frac{1}{\delta}}. \tag{10}$$

The H-distance $d = \rho(\tilde{h}_{t_0}, F^*)$ between the shifted Heaviside step function $\tilde{h}_{t_0}$ and the sigmoidal function F^* satisfies the relation:

$$F^*(t_0 + d) = 1 - \left(1 - \left(\left(1 + \lambda(t_0 + d)^{-\delta}\right)^{-\beta}\right)^{\psi}\right)^{\phi} = 1 - d. \tag{11}$$

The following theorem gives upper and lower bounds for d in the case $\beta = 1$

Theorem 1. *Let*

$$a = -\left(1 - \left(\frac{1}{1 + \left(\left(\frac{-1+(1-0.5^{\frac{1}{\phi}})^{-\frac{1}{\psi}}}{\lambda}\right)^{-\frac{1}{\delta}}\right)^{-\delta}\lambda}\right)^{\psi}\right)^{\phi} \tag{12}$$

$$b = 1 + \delta \left(\left(\frac{-1+(1-0.5^{\frac{1}{\phi}})^{-\frac{1}{\psi}}}{\lambda} \right)^{-\frac{1}{\delta}} \right)^{-1-\delta} \lambda \left(\frac{1}{1+\left(\left(\frac{-1+(1-0.5^{\frac{1}{\phi}})^{-\frac{1}{\psi}}}{\lambda} \right)^{-\frac{1}{\delta}} \right)^{-\delta} \lambda} \right)^{1+\psi} \times$$

$$\times \left(1 - \left(\frac{1}{1+\left(\left(\frac{-1+(1-0.5^{\frac{1}{\phi}})^{-\frac{1}{\psi}}}{\lambda} \right)^{-\frac{1}{\delta}} \right)^{-\delta} \lambda} \right)^{\psi} \right)^{-1+\phi} \phi\psi. \tag{13}$$

The H-distance d between the function $\tilde{h}_{t_0}$ and the function F^* can be expressed in terms of the parameters for $\frac{2b}{-a} > e^2$ as follows:

$$d_l = \frac{1}{\frac{2b}{-a}} < d < \frac{\ln\left(\frac{2b}{-a}\right)}{\frac{2b}{-a}} = d_r. \tag{14}$$

Proof. We define the functions

$$H(d) = F^*(t_0 + d) - 1 + d \tag{15}$$

$$G(d) = a + bd. \tag{16}$$

From Taylor expansion

$$H(d) - G(d) = O(d^2)$$

we see that the function $G(d)$ approximates $H(d)$ with $d \to 0$ as $O(d^2)$ (cf. Fig. 1).

In addition $G'(d) > 0$ and for $\frac{2b}{-a} > e^2$

$$G(d_l) < 0; \quad G(d_r) > 0.$$

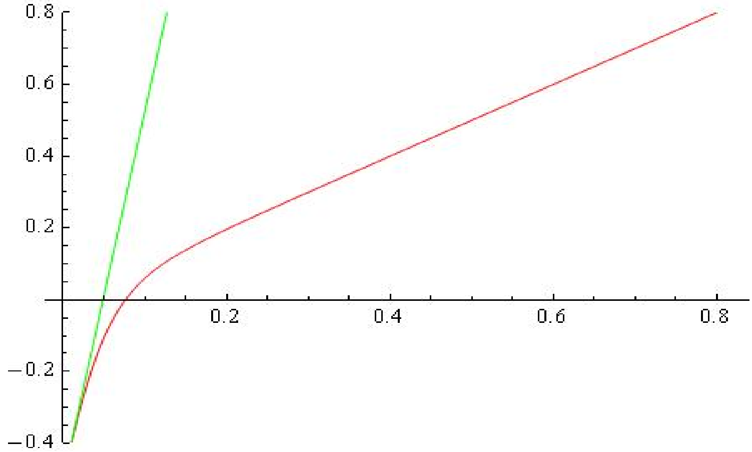

Figure 1: The functions H and G.

This completes the proof of the inequalities (14).

The generated sigmoidal functions $F^*(t)$ for $\lambda = 0.1$; $\delta = 2.5$; $\beta = 1$; $\psi = 0.7$; $\phi = 1.8$ and $\lambda = 0.001$; $\delta = 3.5$; $\beta = 1$; $\psi = 0.8$; $\phi = 1.9$ are visualized on Fig. 2–Fig. 3.

From the Fig. 2–Fig.3 it can be seen that the "supersaturation" is fast.

Following Dagum (1977), in a period when individual data were rarely available, minimized

$$\sum_{i=1}^{n}\left(F_n(t_i) - \left(1-\left(1-\left(\left(1+\lambda t_i^{-\delta}\right)^{-\beta}\right)^{\psi}\right)^{\phi}\right)^2\right).$$

a non–linear least–squares criterion based on the distance between the empirical F_n and the CDF of a Kumaraswamy–Dagum approximation.

The appropriate least–square fitting of the real data (the experimental data - biomass for Xantobacter autotrophycum with electric field, see [26]) by the Dagum model yields for $\beta = 1$, $\lambda = 110$, $\delta = 1.45$, $\psi = 1.35$ and $\phi = 1.1$ and is visualized on Fig. 4.

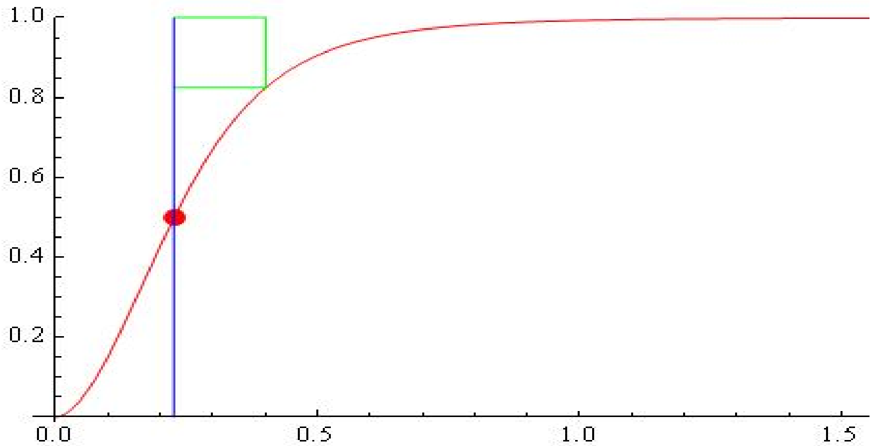

Figure 2: The function $F^*(t)$ for $\lambda = 0.1$; $\delta = 2.5$; $\beta = 1$; $\psi = 0.7$; $\phi = 1.8$; $t_0 = 0.226373$; H-distance $d = 0.175123$; $d_l = 0.0689168$; $d_r = 0.184343$.

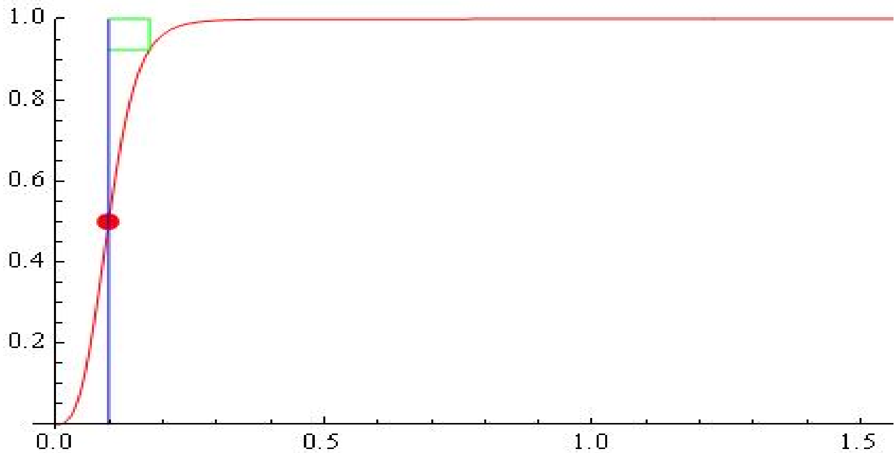

Figure 3: The function $F^*(t)$ for $\lambda = 0.001$; $\delta = 3.5$; $\beta = 1$; $\psi = 0.8$; $\phi = 1.9$; $t_0 = 0.0979526$; H-distance $d = 0.0763243$; $d_l = 0.0244187$; $d_r = 0.0906522$.

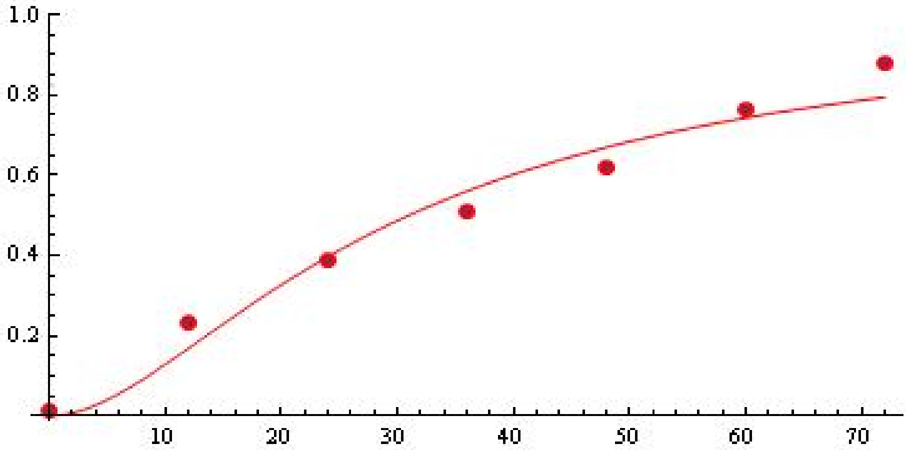

Figure 4: The appropriate least–square fitting of the real data by the Dagum model yields for $\beta = 1$, $\lambda = 110$, $\delta = 1.45$, $\psi = 1.35$ and $\phi = 1.1$.

4. Conclusion

In this paper we prove upper and lower estimates for the Hausdorff approximation of the shifted Heaviside function $\tilde{h}_{t_0}(t)$ by a class of Kumaraswamy–Dagum–Log–Logistic cumulative distribution function – (KD–CDF).

A family of five parametric sigmoidal functions based on Kumuraswamy–Dagum cumulative distribution function is introduced finding application in population dynamics.

Numerical examples, illustrating our results are given.

We propose a software module (intellectual property) within the programming environment *CAS Mathematica* for the analysis of the considered family of (KD–CDF) functions.

For other results, see [14]–[26].

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