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A family of recurrence generated sigmoidal functions based on the Log–logistic function. Some approximation aspects

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Abstract. In this note we construct a family of recurrence generated sigmoidal functions based on the Log–logistic function. The study of some biochemical reactions is linked to a precise Log–logistic function analysis.

We prove estimates for the Hausdorff approximation of the Heaviside step function by means of this family. Numerical examples, illustrating our results are given. The plots are prepared using CAS Mathematica.

Keywords: Log–logistic function · Heaviside step function · Hausdorff distance · Upper and lower bounds.

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1 Introduction

The logistic function belongs to the important class of smooth sigmoidal functions arising from population and cell growth models.

The logistic function was introduced by Pierre François Verhulst [1]–[3], who applied it to human population dynamics. Verhulst proposed his logistic equation to describe the mechanism of the self-limiting growth of a biological population. His equation was rediscovered by A. G. McKendrick [4] for the bacterial growth in broth and was tested using nonlinear parameter identification.

Since then the logistic function finds applications in many scientific fields, including biology, ecology, population dynamics, chemistry, demography, economics, geoscience, mathematical psychology, probability, sociology, political science, financial mathematics, statistics, fuzzy set theory, insurance mathematics, debugging and test theory to name a few [5]–[37].

Logistic functions are also used in artificial neural networks [38]–[46]. Constructive approximation by superposition of sigmoidal functions and the relation with neural networks and radial basis functions approximations is discussed in [42].

Any neural net element computes a linear combination of its input signals, and uses a logistic function to produce the result; often called "activation" function [47]–[50].

The Log–logistic distribution (also known as the Fisk distribution [51]) is a widely used lifetime distribution. For other results, see [53].

The distribution is used to model in fields such as biostatistics, population dynamic, medical research [52] and economics. For the quadratic transmuted family, see Shaw et Buckley [54]. Shaw et al. [54], Gupta et al. [55] study a new model which generalizes the Log–logistic function [56]. For some kinetics interpretation of Log–logistic models, see [57]–[60].

Many biochemical reactions involve binding of a smaller molecule L (called ligand) to a large macromolecule P

$$L + P \longleftrightarrow LP.$$

The oxygen binds to haemoglobin is one of the most important biochemical reactions [60]. The proportion of the bound macromolecules is defined by $\theta = \frac{[LP]}{[P]+[LP]}$.

For arbitrary n from the reaction:

$$nL + P \longleftrightarrow L_n P,$$

the proportion is described by a Hill's formula $\theta = \frac{[L]^n}{K+[L]^n}$, where K and n are empirical parameters. The study of this biochemical reaction is linked to a precise Log–logistic function analysis [61].

Some applications of the Log–logistic and transmuted Log–logistic sigmoids can be found in [61]. Another application area is medicine, where the logistic function is used to model the growth of tumors or to study pharmacokinetic reactions.

2 Preliminaries

Definition 1. *Define the Log–logistic function as*

$$M_0(t) = \frac{t^\beta}{\alpha^\beta + t^\beta} = 1 - \frac{1}{1 + \left(\frac{t}{\alpha}\right)^\beta} \tag{1}$$

where α is a scale parameter.

Evidently, $M_0(\alpha) = \frac{1}{2}$.

Definition 2. *The (basic) step function is:*

$$h_\alpha(t) = \begin{cases} 0, & \text{if} \quad t < \alpha, \\ [0,1], & \text{if} \quad t = \alpha, \\ 1, & \text{if} \quad t > \alpha, \end{cases}$$

usually known as shifted Heaviside step function.

Definition 3. *[62] The Hausdorff distance (the H–distance) [62]* $\rho(f,g)$ *between two interval functions* f, g *on* $\Omega \subseteq \mathbb{R}$*, is the distance between their completed graphs* $F(f)$ *and* $F(g)$ *considered as closed subsets of* $\Omega \times \mathbb{R}$*. More precisely,*

$$\rho(f,g) = \max\{ \sup_{A\in F(f)} \inf_{B\in F(g)} ||A-B||, \sup_{B\in F(g)} \inf_{A\in F(f)} ||A-B||\}, \quad (2)$$

wherein $||.||$ *is any norm in* $\mathbb{R}^2$*, e. g. the maximum norm* $||(t,x)|| = \max\{|t|,|x|\}$*; hence the distance between the points* $A = (t_A, x_A)$*,* $B = (t_B, x_B)$ *in* $\mathbb{R}^2$ *is* $||A-B|| = max(|t_A - t_B|, |x_A - x_B|)$*.*

The Hausdorff approximation of the Heaviside step function by Log–logistic functions of the form (1) is considered in [61] and the following is proved:

Theorem A. [61] *The H-distance* $d_0 = \rho(h_\alpha, M_0)$ *between the Heaviside step function* h_α *and the Log–logistic function* M_0 *can be expressed for* $\frac{\beta}{\alpha} \geq 2$ *as follows:*

$$d_{l_0} = \frac{1}{1+\frac{\beta}{\alpha}} < d_0 < \frac{\ln\left(1+\frac{\beta}{\alpha}\right)}{1+\frac{\beta}{\alpha}} = d_{r_0}, \quad (3)$$

$$\begin{aligned} \tilde{d}_{l_0} \;&=\; \frac{\ln(1+\frac{\beta}{\alpha})}{1+\frac{\beta}{\alpha}} - \frac{\ln\ln(1+\frac{\beta}{\alpha})}{(1+\frac{\beta}{\alpha})\left(1+\frac{1}{\ln(1+\frac{\beta}{\alpha})}\right)} < d_0 < \frac{\ln(1+\frac{\beta}{\alpha})}{1+\frac{\beta}{\alpha}} \\ &+ \frac{\ln\ln(1+\frac{\beta}{\alpha})}{(1+\frac{\beta}{\alpha})\left(\frac{\ln\ln(1+\frac{\beta}{\alpha})}{1-\ln(1+\frac{\beta}{\alpha})} - 1\right)} = \tilde{d}_{r_0}. \end{aligned} \quad (4)$$

3 Main Results

Let us consider the following family of recurrence generated sigmoidal Log–logistic functions

$$M_{i+1}(t) = 1 - \frac{1}{1 + \left(t + \frac{1}{2} - \alpha + M_i(t)\right)^{\beta}}, \quad i = 0, 1, 2, \dots, \tag{5}$$

with

$$M_{i+1}(\alpha) = \frac{1}{2}, \quad i = 0, 1, 2, \dots, \tag{6}$$

based on the Log-logistic function $M_0(t)$.

Let p is the number of recurrences in (5). For $p = 0$, we get the estimates from Theorem A.

The case $p = 1$. In the case of one recursion ($p = 1$, or the same as $i = 0$) from (5)–(6) we get:

$$\begin{aligned} M_1(t) &= 1 - \frac{1}{1+\left(t+\frac{1}{2}-\alpha+M_0(t)\right)^{\beta}} \\ &= 1 - \frac{1}{1+\left(t+\frac{3}{2}-\alpha-\frac{1}{1+\left(\frac{t}{\alpha}\right)^{\beta}}\right)^{\beta}}. \end{aligned} \tag{7}$$

The H-distance $d_1 = \rho(h_\alpha, M_1)$ between the Heaviside step function h_α and the sigmoidal function M_1 satisfies the relation:

$$M_1(\alpha + d_1) = 1 - \frac{1}{1 + \left(d_1 + \frac{3}{2} - \frac{1}{1+\left(1+\frac{d_1}{\alpha}\right)^{\beta}}\right)^{\beta}} = 1 - d_1. \tag{8}$$

The following theorem is valid

Theorem B. *The H-distance d_1 between the function h_α and the function M_1 can be expressed for $\beta\left(1 + \frac{\beta}{4\alpha}\right) \geq 2$ as follows:*

$$d_{l_1} = \frac{1}{1 + \beta + \frac{\beta^2}{4\alpha}} < d_1 < \frac{\ln\left(1 + \beta + \frac{\beta^2}{4\alpha}\right)}{1 + \beta + \frac{\beta^2}{4\alpha}} = d_{r_1}. \tag{9}$$

Proof. From (8) we find

$$\ln\frac{1-d_1}{d_1} = \beta\ln\left(d_1+\frac{3}{2}-\frac{1}{1+\left(1+\frac{d_1}{\alpha}\right)^{\beta}}\right).$$

We examine the function

$$F(d_1) = \beta\ln\left(d_1+\frac{3}{2}-\frac{1}{1+\left(1+\frac{d_1}{\alpha}\right)^{\beta}}\right) - \ln(1-d_1) - \ln\frac{1}{d_1}.$$

From $F'(d_1) > 0$ we conclude that function F is increasing.

Consider the function

$$G(d_1) = \left(1+\beta+\frac{\beta^2}{4\alpha}\right)d_1 - \ln\frac{1}{d_1}. \tag{10}$$

From Taylor expansion we obtain $G(d_1) - F(d_1) = O(d^2)$. Hence $G(d_1)$ approximates $F(d_1)$ with $d_1 \to 0$ as $O(d_1^2)$ (see Fig. 1).

In addition, from

$$G'(d_1) = 1+\beta+\frac{\beta^2}{4\alpha}+\frac{1}{d_1} > 0$$

we conclude that the function G is increasing.

Further, for $\beta\left(1+\frac{\beta}{4\alpha}\right) \geq 2$ we have

$$G\left(\frac{1}{1+\beta+\frac{\beta^2}{4\alpha}}\right) = 1 - \ln\left(1+\beta+\frac{\beta^2}{4\alpha}\right) < 0,$$

$$G\left(\frac{\ln\left(1+\beta+\frac{\beta^2}{4\alpha}\right)}{1+\beta+\frac{\beta^2}{4\alpha}}\right) = \ln\ln\left(1+\beta+\frac{\beta^2}{4\alpha}\right) > 0.$$

This completes the proof of the theorem.

The function M_1 for $\alpha = 0.6$; $\beta = 7$ is visualized on Fig. 2.

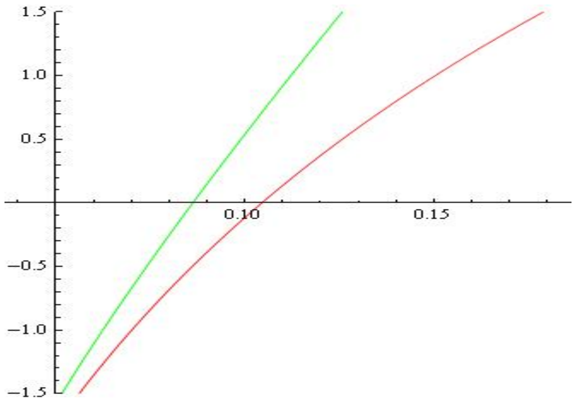

Figure 1: The functions $F(d_1)$ and $G(d_1)$.

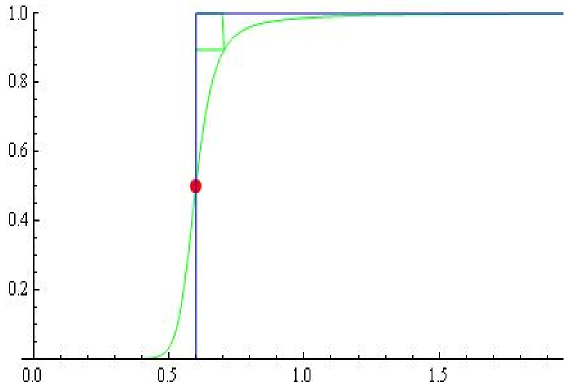

Figure 2: The function M_1) for $\alpha = 0.6$; $\beta = 7$; Hausdorff distance $d_1 = 0.104494$; $d_{l_1} = 0.0351906$; $d_{r_1} = 0.117782$.

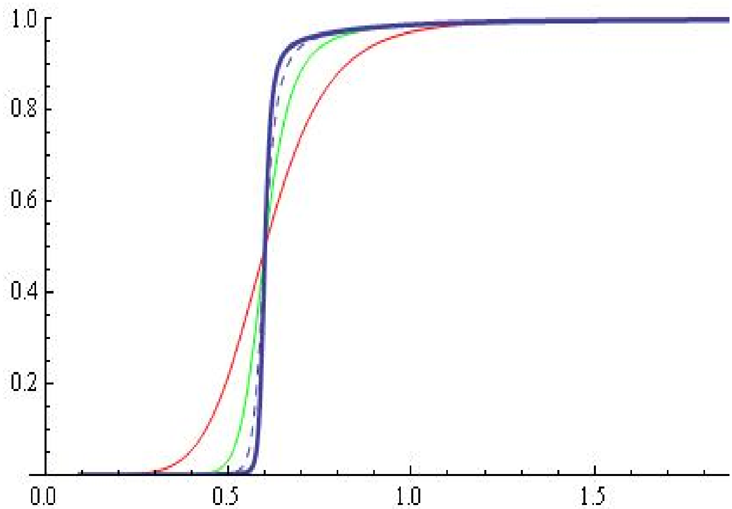

Figure 3: The recurrence generated sigmoidal Log–logistic functions: $M_0(t)$ (red), $M_1(t)$ (green), $M_2(t)$ (dashed) and $M_3(t)$ (thick).

From the nonlinear equation (8) and inequalities (9) we have: $d_1 = 0.104494$; $d_{l_1} = 0.0351906$; $d_{r_1} = 0.117782$.

The recurrence generated sigmoidal Log–logistic functions: $M_0(t)$, $M_1(t)$, $M_2(t)$ and $M_3(t)$ are visualized on Fig. 3

Remark.

For each p, based on the methodology proposed in the present note, the reader may formulate the corresponding approximation problems on his/her own.

3. Conclusions

To achieve our goal, we obtain new estimates for the H-distance between a step function and its best approximating family of recurrence generated sigmoidal Log–logistic functions.

Numerical examples, illustrating our results are given.

We propose a software module (intellectual properties) within the programming environment *CAS Mathematica* for the analysis of the considered family of Log–logistic functions.

The module offers the following possibilities:

- generation of the logistic functions under user defined values of the parameters α and β and number of recursions p;

- calculation of the H-distance $d_p = \rho(h_\alpha, M_p)$, $p = 0, 1, 2, \ldots, p$ between the Heaviside function h_α and the sigmoidal functions M_0, M_1, $M_2, \ldots, M_p$;

- software tools for animation and visualization.

```
Clear[α];
Clear[β];
Manipulate[Dynamic@Show[Plot[f[t], {t, 0, 2}],
    LabelStyle → Directive[Blue, Bold],
    PlotLabel → 1 - 1/(1 + (t + 1/2 - α + 1 - 1/(1 + (t + 1/2 - α + 1 - 1/(1 + (t/α)^β))^β))^β)],
   PlotRange → {Automatic, {0, 1}}],
 {{α, 0.3}, 0.01, 10, Appearance → "Open"},
 {{β, 0.5}, 0.01, 10, Appearance → "Open"},
 Initialization :→
  (f[t_] := 1 - 1/(1 + (t + 1/2 - α + 1 - 1/(1 + (t + 1/2 - α + 1 - 1/(1 + (t/α)^β))^β))^β))]
```

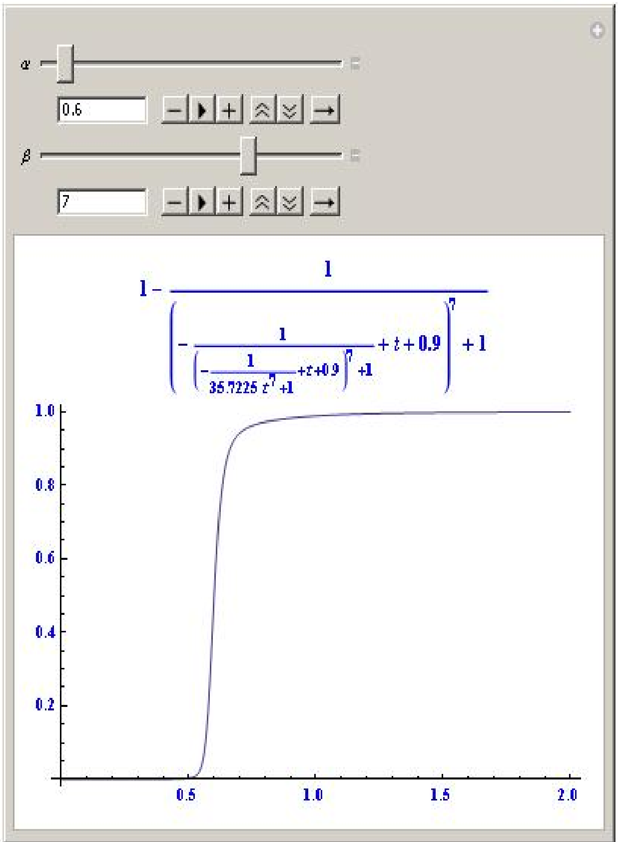

Figure 4: An example of the usage of dynamical and graphical representation for the family $M_{i+1}(t)$. For example $p = 2; i = 1$, $\alpha = 0.6$, $\beta = 7$; Hausdorff distance $d_2 = 0.073621$. The plots are prepared using CAS Mathematica.

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