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Investigations on the G Family with Baseline Burr XII Cumulative Sigmoid

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Abstract. In this paper we study the one–sided Hausdorff approximation of the shifted Heaviside step function by a class of the Zubair–G family of cumulative lifetime distribution with baseline Burr XII c.d.f. The estimates of the value of the best Hausdorff approximation obtained in this article can be used in practice as one possible additional criterion in "saturation" study.

As an illustrative examples we consider the fitting of the new model using experimental oil palm data [1], [2] and "cancer data" [21], [22].

Numerical examples, illustrating our results are presented using programming environment *CAS Mathematica*.

AMS Subject Classification: 68N30, 41A46

Keywords: Zubair–G family of cumulative lifetime distribution, Zubair–G Family with baseline Burr XII (cdf), Heaviside function, Hausdorff approximation, upper and lower bounds

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1 Introduction

In [3], a new family of lifetime distributions, called the Zubair–G family of distributions is introduces.

The new family is defined by the following cumulative distribution function (cdf)

$$F(t;\lambda) = \frac{e^{\lambda G^2(t)} - 1}{e^{\lambda} - 1}, \tag{1}$$

where $\lambda > 0$.

If, G is the (cdf) of the baseline model, then the distribution function (1) will be the (cdf) of the Zubair–G family.

Some comments on a Zubair-G Family of cumulative lifetime distributions with baseline Weibull (cdf) can be found in [4].

For example, if $G(t)$ be (cdf) of the Burr XII distribution given by

$$G(t) = 1 - \frac{1}{\left(1 + \left(\frac{t}{b}\right)^c\right)^k} \tag{2}$$

then the (cdf) of the Z–Burr distribution has the form

$$F(t) = \frac{e^{\lambda\left(1 - \frac{1}{\left(1+\left(\frac{t}{b}\right)^c\right)^k}\right)^2} - 1}{e^{\lambda} - 1} \tag{3}$$

where $a > 0$ and $b > 0$.

We consider the following class of this family with application to the population dynamics and debugging theory:

$$M(t) = \frac{e^{\lambda\left(1 - \frac{1}{\left(1+\left(\frac{t}{b}\right)^c\right)^k}\right)^2} - 1}{e^{\lambda} - 1}, \tag{4}$$

with

$$t_0 = b\left(\left(\frac{1}{1-\sqrt{\frac{1}{\lambda}\ln\frac{e^{\lambda}+1}{2}}}\right)^{\frac{1}{k}}-1\right)^{\frac{1}{c}}; \qquad M(t_0)=\frac{1}{2}. \tag{5}$$

In this note we study the Hausdorff approximation of the *shifted Heaviside step function*

$$h_{t_0}(t) = \begin{cases} 0, & \text{if} \quad t < t_0, \\ [0,1], & \text{if} \quad t = t_0, \\ 1, & \text{if} \quad t > t_0 \end{cases}$$

by this family.

Definition 1. *[5] The Hausdorff distance (the H–distance) $\rho(f,g)$ between two interval functions f, g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,*

$$\rho(f,g) = \max\{\sup_{A\in F(f)} \inf_{B\in F(g)} ||A-B||, \sup_{B\in F(g)} \inf_{A\in F(f)} ||A-B||\},$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t,x)|| = \max\{|t|,|x|\}$; hence the distance between the points $A=(t_A,x_A)$, $B=(t_B,x_B)$ in $\mathbb{R}^2$ is $||A-B|| = max(|t_A-t_B|, |x_A-x_B|)$.

As an illustrative example we consider the fitting the new model against experimental oil palm data [1], [2].

2 Main Results

The one–sided Hausdorff distance d between the function $h_{t_0}(t)$ and the sigmoid - ((4)–(5)) satisfies the relation

$$M(t_0+d) = 1-d. \tag{6}$$

The following theorem gives upper and lower bounds for d

Theorem. Let

$$p = -\tfrac{1}{2},$$

$$q = 1 + \frac{ck\lambda(1+e^{\lambda})\sqrt{\frac{1}{\lambda}\ln\frac{e^{\lambda}+1}{2}}\left(1-\sqrt{\frac{1}{\lambda}\ln\frac{e^{\lambda}+1}{2}}\right)^{\frac{k+1}{k}}}{b(e^{\lambda}-1)} \times \left(\left(\frac{1}{1-\sqrt{\frac{1}{\lambda}\ln\frac{e^{\lambda}+1}{2}}}\right)^{\frac{1}{k}} - 1\right)^{\frac{c-1}{c}}, \tag{7}$$

$$r = 2.1q.$$

For the one–sided Hausdorff distance d between $h_{t_0}(t)$ and the sigmoid ((4)–(5)) the following inequalities hold for: $q > \frac{e^{1.05}}{2.1}$

$$d_l = \frac{1}{r} < d < \frac{\ln r}{r} = d_r. \tag{8}$$

Proof. Let us examine the function:

$$F(d) = M(t_0 + d) - 1 + d. \tag{9}$$

From $F'(d) > 0$ we conclude that function F is increasing.

Consider the function

$$G(d) = p + qd. \tag{10}$$

From Taylor expansion we obtain $G(d) - F(d) = O(d^2)$.

Hence $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see Fig. 1). In addition $G'(d) > 0$.

Further, for $q > \frac{e^{1.05}}{2.1}$ we have $G(d_l) < 0$ and $G(d_r) > 0$.

This completes the proof of the theorem.

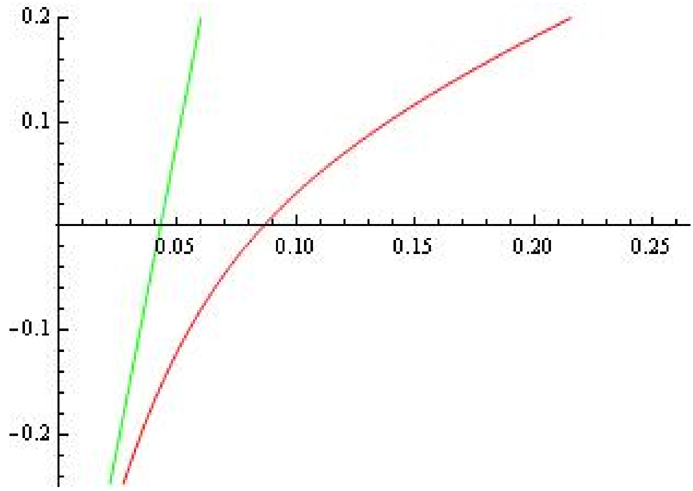

Figure 1: The functions $F(d)$ and $G(d)$ for $b = 0.05$; $c = 1.9$; $k = 1.6$; $\lambda = 0.9$.

3 Numerical examples.

The model ((4)–(5)) for $b = 0.05$; $c = 1.9$; $k = 1.6$; $\lambda = 0.9$, $t_0 = 0.0635798$ is visualized on Fig. 2.

From the nonlinear equation (6) and inequalities (8) we have: $d = 0.0860937$, $d_l = 0.0406867$, $d_r = 0.130273$.

The model ((4)–(5)) for $b = 0.03$; $c = 1.91$; $k = 1.5$; $\lambda = 0.85$, $t_0 = 0.0398088$ is visualized on Fig. 3.

From the nonlinear equation (6) and inequalities (8) we have: $d = 0.0668717$, $d_l = 0.0268802$, $d_r = 0.0972085$.

The model ((4)–(5)) for $b = 0.01$; $c = 1.95$; $k = 2$; $\lambda = 0.95$, $t_0 = 0.0107492$ is visualized on Fig. 4.

From the nonlinear equation (6) and inequalities (8) we have: $d = 0.0228$, $d_l = 0.00664579$, $d_r = 0.0333205$.

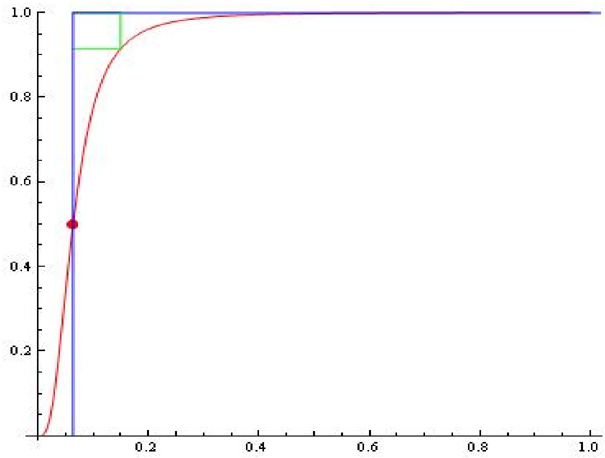

Figure 2: The model ((4)–(5)) for $b = 0.05$; $c = 1.9$; $k = 1.6$; $\lambda = 0.9$, $t_0 = 0.0635798$; H–distance $d = 0.0860937$, $d_l = 0.0406867$, $d_r = 0.130273$.

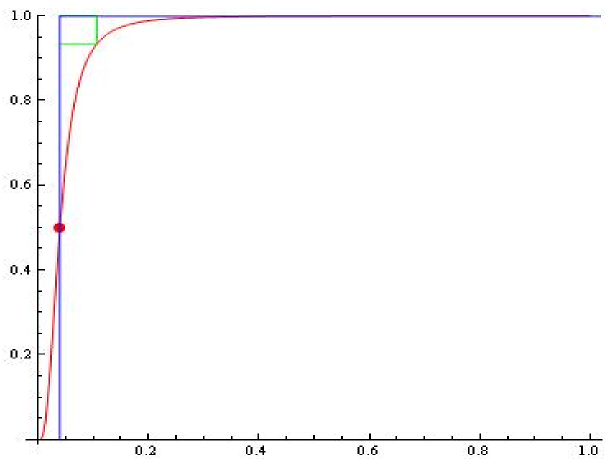

Figure 3: The model ((4)–(5)) for $b = 0.03$; $c = 1.91$; $k = 1.5$; $\lambda = 0.85$, $t_0 = 0.0398088$; H–distance $d = 0.0668717$, $d_l = 0.0268802$, $d_r = 0.0972085$.

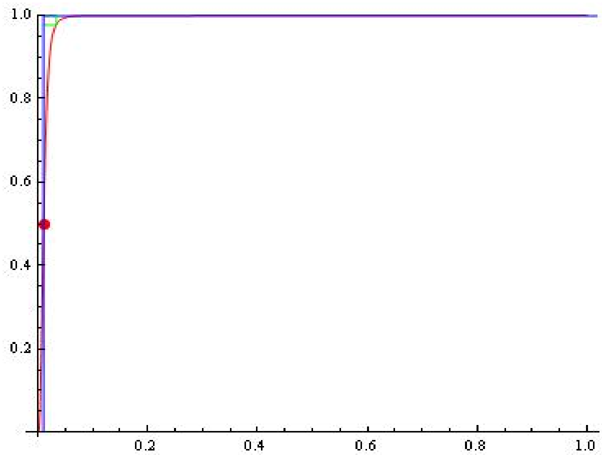

Figure 4: The model ((4)–(5)) for $b = 0.01$; $c = 1.95$; $k = 2$; $\lambda = 0.95$, $t_0 = 0.0107492$; H–distance $d = 0.0228$, $d_l = 0.00664579$, $d_r = 0.0333205$.

From the above examples, it can be seen that the proven estimates (see Theorem) for the value of the Hausdorff approximation is reliable when assessing the important characteristic - "saturation".

4 Applications

4.1 Population Dynamics

Consider the model:

$$M(t) = \omega \frac{e^{\lambda\left(1-\frac{1}{\left(1+\left(\frac{t}{b}\right)^c\right)^k}\right)^2} - 1}{e^{\lambda} - 1}. \qquad (11)$$

The model (11) based on the data of Table 1 for the estimated

$Year$	$Weight$	$The\, appropriate\, fitting\, by\, function$ (11)
4	11.78	11.6502
5	18.43	18.6774
6	25.21	25.2094
7	30.78	30.2353
8	33.03	33.6174
9	35.66	35.7011
10	36.96	36.9164
11	37.97	37.6026
12	38.04	37.9832
13	39.20	38.1926
14	36.50	38.3075
15	37.21	38.3706
16	39.97	38.4054
17	38.45	38.4248

Table 1: The oil palm yield data [1], [2]

parameters:

$$\omega = 38.45;\ b = 15.765;\ c = 1.53476;\ k = 11.8289;\ \lambda = 2.12748$$

is plotted on Fig. 5.

For the predictive power (PP) criterion:

$$PP = \sum_{i=4}^{17} \left(\frac{M(t_i) - y_i}{y_i} \right)^2$$

we find $PP = 0.0066476$.

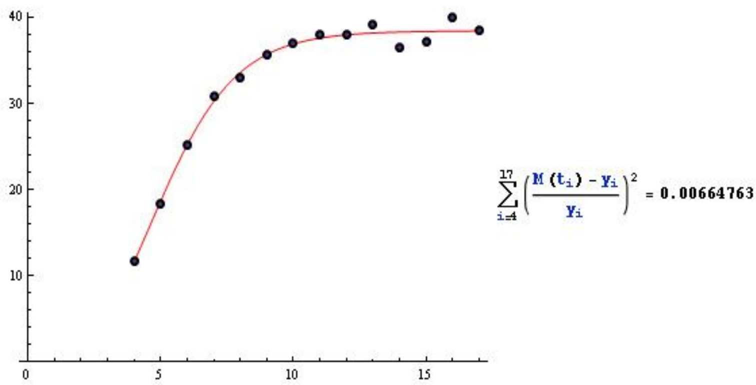

Figure 5: The model $M(t)$ based on the data of Table 1.

4.2 Tumor Growth

Here we give an application of the new cumulative sigmoid for analysis of the following "cancer data" (for some details see, [21], [22]).

days	4	7	10	12	14	17	19	21
$R(t)$	0.415	0.794	1.001	1.102	1.192	1.22	1.241	1.3

Table 2: The "cancer data" [21], [22].

Consider the model (11) based on the data of Table 2 for the estimated parameters:

$$\omega = 1.39576;\ c = 1.45283;\ b = 4.45283;\ k = 1.45079;\ \lambda = 0.475223$$

is plotted on Fig. 6.

For the predictive power (PP) criterion:

$$PP = \sum_i \left(\frac{M(t_i) - y_i}{y_i}\right)^2$$

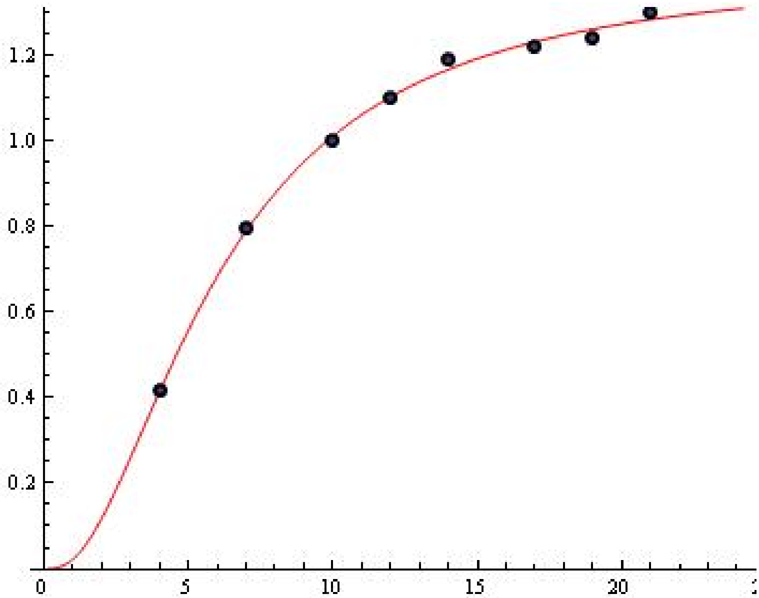

Figure 6: The model $M(t)$ based on the "cancer data".

we find $PP = 0.00107871$.

From the conducted experiments it can be concluded that the examined model can be successfully used in the field of Population dynamics.

For some approximation, computational and modelling aspects, see [6]–[17].

Some software reliability models, can be found in [18]–[20].

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