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60th Anniversary of the Seminal Paper on Interval Analysis and Computations by T. Sunaga

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Sunaga considered all computational procedures, which had been traditionally defined on real numbers, as being too ideal and proposed to replace them by the procedures on real intervals in order to make everything "more realistic". Sunaga studied many different kinds of numerical procedures including the Taylor-series interval solution of the initial-value problem of ordinary differential equations.

M. Iri [11]

The purpose of the present note is to mark the 60-th anniversary of the publication of the seminal work by the Japan mathematician Teruo Sunaga [30]. The paper summarizes the results of his Master Thesis [29]. Sunaga's work sets the foundation of the contemporary interval analysis and reliable computing. This is an interdisciplinary

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Prof. Teruo Sunaga (1929-1995)

field, combining abstract mathematical theories and practical applications related to computer science, numerical analysis and mathematical modeling in the natural, engineering and social-economic sciences.

At present there exist hundreds of related publications including many monographs, conference proceedings and collections of articles, numerous related international meetings have been organized. It is a commonplace view that interval analysis will continue to play a significant role in applied mathematics and science.

Sunaga's work is an excellent example for a successful combination of abstract mathematics with applied computational science. Both aspects are based on a firm algebraic foundation comprising the interval arithmetic operations together with the interval lattice operations. From this fundament the author develops the applied computational aspect by adding the directed rounding to the arithmetic operations in order to achieve guaranteed inclusions and thus verified computational results. The author then demonstrates the power of the interval verified computations on the most popular examples of numerical procedures such as Taylor expansion, Newton method for solving algebraic equation and Euler's method for solving ordinary differential equations.

A study performed on the occasion of the 40-th anniversary of the appearance of Sunaga's work [20] summarizes the most important achievements of the author. Now, twenty years later, it is again time to recall this extraordinary work and to trace its impact on the mathematical and computational sciences worldwide. Let us recall that shortly after publication Sunaga's paper came to the attention of the eminent American mathematician, numerical analyst, George Forsythe and his doctoral student R. Moore [26]. In his doctoral dissertation [21] R. Moore further develops the interval verification methods. Moore made his best for the promotion of the interval methods via numerous publications and conference proceedings [23]. Especially popular is Moore's monograph [22], which triggered an enormous activity in the field of interval computations. An important result of the joint efforts of many specialists in this direction was the establishing of an IEEE standard for interval arithmetic [10]. Numerous computational platforms using interval arithmetic and directed rounding were developed. Let us mention one of the most popular platform for interval computations INTLAB [27]. Nowadays a number of Japan mathematicians work in the field of interval analysis. The last meeting of the well-established SCAN series of international conferences was organized in Tokyo [28].

In the last part of his work Sunaga focuses on the need of further theoretical developments of his interval algebra in direction topological group structures. Let us note that an embedding of the interval arithmetic semigroup system into complete algebraic structures with group properties is trivial for the operation interval addition, but is quite nontrivial for interval multiplication under preserving the lattice relations. An inclusion invariant interval multiplication was formulated and studied by E. Kaucher [12], [13], which opened the way to an wide field of interval algebra research known as Kaucher interval arithmetic.

The ideas of Sunaga and Kaucher have produced their strong impact on the authors of this note and on many colleagues and collaborators wold-wide, see e.g. [1]– [7], [24]–[25], [14]–[18], [8].

It is now impossible to enlist the many scientific directions that have been already influenced by the developments in interval analysis and reliable computing, nevertheless we are tempted to mention some of the rapidly developed areas. We wish to note the ongoing penetration of the interval ideas in the field of life sciences, such as biology, medicine, ecology, engineering and socio-economic sciences. Interval methods are very appropriate for studying biological processes [19]. The number of interval-related papers in life sciences is growing, for a sample of paper titles see the last part of the reference list [31]–[40].

Novices in the field of interval computations and interval analysis may take a look at the website [9] containing useful information about the history and the state of art in the field. The website is developed and maintained by Prof. Vladik Kreinovich.

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