Article Biomath Communications 5 (2018)

Biomath Communications

www.biomathforum.org/biomath/index.php/conference

Intervals and (non-)negative numbers

Svetoslav M. Markov
Institute of Mathematics and Informatics,
Bulgarian Academy of Sciences

Abstract

Intervals have a double nature: they can be considered as compact sets of real numbers (set-intervals) or as approximate numbers. A set-interval is presented as an ordered pair of two real numbers (interval end-points), whereas an approximate number is an ordered pair consisting of a real "exact" number and a nonnegative error bound. Thus, differently to the case with set-intervals, where both endpoints are real numbers, when operating with approximate numbers, one should know the algebraic properties of the arithmetic operations over error bounds, that is over nonnegative numbers. This work is devoted to the algebraic study of the arithmetic operations addition and multiplication by scalars for approximate numbers, resp. for errors bounds. Such a setting leads to so-called quasilinear spaces. We formulate and prove several new properties of such spaces, which are important from computational aspect. In particular, we focus our study on the operation "distance between two nonnegative numbers". We show that this operation plays an important role in the study of the concept of linear independence of interval vectors, the latter being correctly defined.

Citation: Svetoslav Markov, Intervals and (non-)negative numbers, Biomath Communications 5, pp. 114-135, https://doi.org/10.11145/bmc.2018.12.317

1 Introduction

Interval analysis is now an established sub-domain of numerical analysis. In its contemporary form it starts with the work of T. Sunaga [7], [10]. Related key words are: "interval arithmetic", "interval computations", "reliable computing", etc. The field has a journal [12] and mailing-list [13] comprising several hundred users. Biannual conferences are organized the last one (12th in the series) taking place in Lyon, France, in 2010, see: `http://scan2010.ens-lyon.fr/`. A well organized and maintained website is [14]. A nice popular introduction in interval analysis (and related issues) is the article [3]. Currently an IEEE P1788 Working Group develops a standard for interval arithmetic [2].

In this work we study some properties of the interval arithmetic operations for addition/subtraction and multiplication by scalars, remaining in the domain of proper intervals. We focus on the inner interval arithmetic operations for adition/substraction. Recall that the group analogue of such a setting has been already well explored, leading to so-called quasivector space, see [8]. In a quasivector space of group structure linear independence of interval vectors is introduced in a natural way. Here we show that in a quasilinear space of monoidal structure linear independence of interval vectors can also be defined using inner interval operations. This has been done in this work for first time.

2 Preliminaries

By a one-dimensional interval we shall mean a compact set on the real line. Intervals have a double nature: they can be interpreted either as sets of real numbers (set-intervals) or as approximate numbers. Set-intervals are presented as ordered pairs of two real numbers interpreted as end-points. The interpretation of intervals as *set-intervals* is especially useful in the area of global optimization, where one often needs to work with large intervals. Alternatively, intervals can be viewed as *approximate numbers*, which are ordered pairs consisting of a real number, interpreted as "main" value (sometimes considered as "mean", "probable", "highly possible", "true", etc.) and a non-negative real number interpreted as an "error bound". The "approximate number"-concept excludes the consideration of large intervals; in praxis it also makes no sense to consider approximate numbers containing zero.

2.1 Interval arithmetic in "set-interval" notation

We first introduce interval arithmetic operations for addition/subtraction ("+", "−", etc.) and multiplication by scalars "$*$", as well as the inclusion relation relation "$\subseteq$". We shall formulate these operations using both the "set-" and the "approximate number"-concepts.

Denote the set of reals by $\mathbb{R}$ and the set of all real compact intervals by $\mathbf{I}\mathbb{R}$. Given the endpoints $\underline{a},\ \overline{a} \in \mathbb{R},\ \ \underline{a} \leq \overline{a}$, denote $[\underline{a},\ \overline{a}] = \{x \mid \underline{a} \leq x \leq \overline{a}\}$. For two intervals $A = [\underline{a},\ \overline{a}]$, $B = [\underline{b},\ \overline{b}] \in \mathbf{I}\mathbb{R}$ we have:

$$\begin{aligned}
[\underline{a},\overline{a}] \ + \ [\underline{b},\overline{b}] &= [\underline{a}+\underline{b},\ \overline{a}+\overline{b}],\\
[\underline{a},\overline{a}] \ - \ [\underline{b},\overline{b}] &= [\underline{a}-\overline{b},\ \overline{a}-\underline{b}],\\
[\underline{a},\ \overline{a}] +^- [\underline{b},\ \overline{b}] &= [\underline{a}+\underline{b} \vee\ \overline{a}+\overline{b}],\\
[\underline{a},\ \overline{a}] -^- [\underline{b},\ \overline{b}] &= [\underline{a}-\underline{b} \vee\ \overline{a}-\overline{b}],\\
\alpha * [\underline{b},\ \overline{b}] &= [\alpha\underline{b} \vee\ \alpha\overline{b}],\\
[\underline{a},\overline{a}] \ \subseteq \ [\underline{b},\overline{b}] &= \underline{b} \leq \underline{a} \ \&\ \overline{a} \leq \overline{b}.
\end{aligned}$$

The notation $[\alpha \vee \beta]$ means the (interval) set of all reals between α and β; this notation is useful whenever one does not know whether $\alpha \leq \beta$ or $\alpha > \beta$.

The operations "+", "−" will be referred as outer addition/subtraction, whereas the operations "$+^-$", "$-^-$" will be referred as inner addition/subtraction [5].

2.2 Interval arithmetic in "approximate-number" notation

(Narrow) intervals can be interpreted as approximate numbers; such are for example floating point numbers. An approximate number is an ordered pair consisting of a real number considered as "exact" and an error bound. In the case of floating-point numbers the "exact" real number is a machine number and an error bound is, e. g. the distance between the neighboring two machine numbers. Error bounds, sometimes called computational errors or just *errors*, are (real) nonnegative numbers.

Denote the set of nonnegative reals by $\mathbb{R}^+ = \{a \in \mathbb{R} \mid a \geq 0\}$. Given two intervals $a' \in \mathbb{R},\ a'' \in \mathbb{R}^+$, denote $(a'; a'') = \{x \mid |x - a'| \leq a''\}$.

Given $(a'; a'')$, $(b'; b'') \in \mathbf{I}\mathbb{R}$ with $a', b' \in \mathbb{R},\ a'', b'' \in \mathbb{R}^+$, we have:

$$\begin{array}{rcl}(a'; a'') \ + \ (b'; b'') & = & (a' + b';\ a'' + b''), \\ (a'; a'') \ - \ (b'; b'') & = & (a' - b';\ a'' + b''), \\ (a'; a'') +^- (b'; b'') & = & (a' + b';\ |a'' - b''|), \\ (a'; a'') -^- (b'; b'') & = & (a' - b';\ |a'' - b''|), \\ \alpha * (b'; b'') & = & (\alpha b';\ |\alpha| b''), \\ (a'; a'') \ \subseteq \ (b'; b'') & = & |b' - a'| \leq b'' - a''.\end{array}$$

We note that in the above definition of the operations "$+^-$", "$-^-$" the expression $|a'' - b''|$ appears which is the "distance" between the nonnegative numbers $|a''$ and $b''|$. We next show that this "distance" defines a natural operation arising in the additive set of real numbers and shall explore its algebraic properties. To this end we next recall the operation addition of real numbers.

2.3 Interval arithmetic and functional ranges

To have an idea of the utilization of the interval arithmetic operations, we briefly discuss their relation to functional ranges.

Let f, g be two continuous functions defined on the interval $X \in I(\mathbb{R})$. The ranges of f, g are $f(X) = \{f(x) \mid x \in X\}$, resp. $g(X) = \{g(x) \mid x \in X\}$. Assume that these ranges are known, we want to find out the range of the sum $f + g$. We can easily see that for the range of the sum $f + g$ we have the inclusion: $(f + g)(X) \subseteq f(X) + g(X)$, where "+" is the (outer) interval addition. However, we would like to find out a more "sharp" relation, possibly equality relation. To this end we note that in the special case when f, g are equally monotone functions we have: $(f + g)(X) = f(X) + g(X)$. We also note that the relation $(f + g)(X) = f(X) + g(X)$ is true for any two equally monotone functions f, g. Hence this relation can be used to define the operation (outer) addition of intervals as follows.

Definition EM. Given $A, B \in I(\mathbb{R})$ take any two *equally monotone* functions f, g such that $f(X) = A, \;\; g(X) = B$. Then $(f+g)(X)$ depends only on the choice of A, B. We thus define (outer) addition of A, B by means of the relation: $A + B = (f + g)(X)$.

Note that for "smooth" functions and "narrow" intervals X "half" of the practical situations are such that functions f, g are equally monotone functions. In the other "half" of the situations the functions f, g are differently monotone functions on X.

Note also that in practice, monotonicity is a rather weak restriction — because if X is small enough, we usually deal with functions f, g that are monotone on X.

Now let us assume that f, g are *differently* monotone functions on X? In this case we have of course $(f+g)(X) \subseteq f(X)+g(X)$ but this inclusion could be very "rough". For example: $\{x + (-x) \mid x \in X\} \subseteq X + (-X) = X - X$. Note that $\omega(X - X) = 2\omega(X)$.

Definition DM. Given $A, B \in I(\mathbb{R})$ take any two *differently monotone* functions f, g such that the sum $f + g$ is monotone and $f(X) = A, \;\; g(X) = B$. Note that the interval range $(f + g)(X)$ depends only on the choice of A, B. Thus we define the operation: $A +^{-} B = (f + g)(X)$ and call this operation "inner addition" of the intervals $A, B \in I(\mathbb{R})$.

Propositions for the computation of ranges can now be formulated, such as:

$$(f+g)(X) = \begin{cases} f(X) + g(X) & \text{if } f, g \text{ are equally monotone;} \\ f(X) +^- g(X) & \text{if } f, g \text{ are differently monotone,} \end{cases}$$

etc. The above considerations demonstrate the use of interval operations. For more results within these lines consult [6].

3 Intervals as approximate numbers

To compute with approximate numbers one should know the arithmetic operations on non-negative numbers and the properties of these operations. Such computations require suitable definitions and study of the arithmetic operations and order relations over the set of non-negative numbers. In the sequel we discuss the algebraic properties of non-negative numbers starting from familiar properties of real numbers. We restrict ourselves in the algebraic study of the arithmetic operations addition and multiplication by scalars for non-negative numbers. Such a setting leads to so-called quasilinear (interval) spaces. In particular, we focus our study on the operation "$+^-$" defined as the distance between two nonnegative real numbers: $A +^- B = |A - B|$ in combination with the familiar order relation. This operation plays an important role in the computation with error bounds and approximate numbers. We study the algebraic properties of this operation. Based on this study we formulate and prove some new algebraic properties of non-negative numbers, which are important from computational aspect.

For simplicity we start with enlisting the algebraic properties of the familiar system $(\mathbb{R}, +, \leq)$ involving the set of real numbers together with the arithmetic operation addition "$+$" and the order relation preceding "$\leq$". Parts of the material in this section is developed in detail in [9].

3.1 Algebraic properties of $(\mathbb{R}, +, \leq)$

We recall briefly the algebraic properties of real numbers with respect to addition. As we know $(\mathbb{R}, +)$ is an additive group, that is

i) "+" is a closed (total) operation;
ii) "+" is associative: $(a + b) + c = a + (b + c)$;
iii) there is an identity (null) element 0, such that $a + 0 = a$ for all a;
iv) for every a there exists an additive inverse (opposite) element $-a$, such that $a + (-a) = 0$.

Property iv) induces operation subtraction $a - b = a + (-b)$ and, consequently, the property subtractability, in the sense that equation $a + x = b$ has an unique solution for all $a, b \in \mathbb{R}$, namely $x = b - a = b + (-a)$.

Using algebraic terminology we can say: due to property i) $(\mathbb{R}, +)$ is a magma; due to properties i)–ii) $(\mathbb{R}, +)$ is a semigroup; due to i)–iii) $(\mathbb{R}, +)$ is a monoid; and due to i)–iv) $(\mathbb{R}, +)$ is a group. Every group obeys also property:

v) cancellation law: $a + x = b + x \Longrightarrow a = b$.

An algebraic system may also satisfy:

vi) commutative law: $a + b = b + a$.

The additive system of reals $(\mathbb{R}, +)$ satisfies all enlisted properties i)– vi) and thus is a commutative (abelian) group.

Order isotonicity. In system $(\mathbb{R}, +, \leq)$ the preceding order "$\leq$" is consistent with addition, in the sense that for $a, b, c \in \mathbb{R}$ we have $a \leq b \implies a + c \leq b + c$. As a consequence we have for $a, b, c, d \in \mathbb{R}$: $a \leq b,\ c \leq d \implies a + c \leq b + d$.

Inverse isotonicity of addition. If $a, b, c \in \mathbb{R}$, then $a + c \leq b + c \Longrightarrow a \leq b$, in particular: $a + c = b + c \Longrightarrow a = b$ (cancellation law).

3.2 Real numbers in signed-magnitude form

Denote by $\mathbb{R}^+ = \{a \in \mathbb{R} \mid a \geq 0\}$ the set of non-negative real numbers and let $\Lambda = \{+, -\}$. A real number $a \in \mathbb{R}$ is usually presented in the form $\pm A$, that is as an ordered pair of the form $(A; \alpha)$, with $A = |a| \in \mathbb{R}^+$ and $\alpha = \sigma(a) \in \Lambda$, where

$$\sigma(a) = \begin{cases} + & \text{if } \; a \geq 0; \\ - & \text{if } \; a < 0. \end{cases}$$

We have

$$a = (A; \alpha) \in \{(X; \xi) \mid X \in \mathbb{R}^+, \; \xi \in \Lambda\} = \mathbb{R}^+ \otimes \Lambda.$$

When computing with real numbers we usually use the above presentation $a = (A; \alpha)$ which will be further referred as signed-magnitude form, briefly *sm-form*. Practically this means that we perform some operations separately on the nonnegative component (magnitude) $A \in \mathbb{R}^+$ and on the sign α. Thus we have to know the algebraic properties of nonnegative real numbers, in particular those of the additive system $(\mathbb{R}^+, +)$. In computational sciences nonnegative real numbers are often related to computational errors (error bounds); thus instead of "nonnegative real numbers" we shall sometimes speak of "errors", "error numbers" or briefly "e-numbers".

An important difference between $\mathbb{R}$ and $\mathbb{R}^+$ with respect to addition is that $\mathbb{R}$ is an additive group whereas $\mathbb{R}^+$ is a semigroup. There are no inverse elements in $(\mathbb{R}^+, +)$; consequently no operation subtraction and generally no solution to an equation of the form $a + x = b$. To underline this difference in the sequel we shall denote the elements of $\mathbb{R}$ by lower-case letters $a, b, c, ...$, whereas the elements of $\mathbb{R}^+$ by upper-case letters, $A, B, C, ...$

The set of pairs $\mathbb{R}^+ \otimes \Lambda$ admits both elements $(0; +)$ and $(0; -)$, which both correspond to the element $0 \in \mathbb{R}$. Assuming $(0; +) = (0; -)$, we obtain a bijection between $\mathbb{R}$ and $\mathbb{R}^+ \otimes \Lambda$. This allows us to identify a real number with its sm-form $a = (A; \alpha)$.

3.3 Addition of reals in signed-magnitude form

Let us formulate addition of real numbers using the sm-form $a = (A;\alpha)$ minding the isomorphism $(\mathbb{R}, +, \leq) \cong (\mathbb{R}^+ \otimes \Lambda, +, \leq)$. Since addition of real numbers with the same sign and with a different sign are handled differently, to add $(A;\alpha),\ (B;\beta) \in \mathbb{R}^+ \otimes \Lambda$ we consider separately the cases $\alpha = \beta$ and $\alpha \neq \beta$. In the case $\alpha = \beta$ we have $(A;\alpha) + (B;\alpha) = (A + B;\alpha)$. Here "$A + B$" is the operation addition in $\mathbb{R}^+$ which is the restriction of addition in $\mathbb{R}$. For simplicity we use same notation for addition and order both in $\mathbb{R}$ and $\mathbb{R}^+$.

To add $(A;\alpha),\ (B;\beta) \in \mathbb{R}^+ \otimes \Lambda$ in the case $\alpha \neq \beta$ we need the operation $|A - B|$ in $\mathbb{R}^+$. Since there is no subtraction in $\mathbb{R}^+$ we shall denote $A +^- B = |A - B|$ and define operation "$+^-$" correctly as follows:

Definition 1. C-addition of $A, B \in \mathbb{R}^+$ is defined by

$$A +^- B = \begin{cases} Y|_{B+Y=A} & \text{if } B \leq A; \\ X|_{A+X=B} & \text{if } A \leq B. \end{cases} \tag{1}$$

Note that if both solutions X, Y exist in (1) (which only happens when $A = B$), then they coincide and $X = Y = 0$. If $A \neq B$ then exactly one of the equations $A + X = B,\ B + Y = A$ is solvable. Operation (1) is well defined in $\mathbb{R}^+$; we call it "c-addition" ("c" stands for "conditional").

Define a mapping $\mu : \mathbb{R}^+ \otimes \Lambda^2 \longrightarrow \Lambda$ as follows:

$$\mu((A;\alpha),(B;\beta)) = \begin{cases} \alpha & \text{if } B \leq A, \\ \beta & \text{if } B > A. \end{cases}$$

In the case $\alpha \neq \beta$ we have $(A;\alpha) + (B;\beta) = (A +^- B;\ \mu(a,b))$. Summarizing, we have

$$a + b = (A;\alpha) + (B;\beta) = \begin{cases} (A + B;\alpha) & \text{if } \alpha = \beta; \\ (A +^- B;\ \mu(a,b)) & \text{if } \alpha \neq \beta, \end{cases}$$

which can be compactly written as

$$(A;\alpha) + (B;\beta) = (A +^{\alpha\beta} B;\ \mu(a,b)). \tag{2}$$

In (2) we assume that for $\alpha, \beta \in \Lambda$ a binary boolean operation "$\cdot$" is defined by $\alpha \cdot \beta = \alpha\beta = \{+, \alpha = \beta;\ -, \alpha \neq \beta\}$. In addition we assume $+^+ = +$.

Formula (2) shows that addition of two real numbers in sm-form induces the operation c-addition in the set of nonnegative reals. C-addition is defined as solution of an algebraic equation of the form $A + X = B$ and therefore c-addition appears naturally in $\mathbb{R}^+$ in the same manner as subtraction appears in $\mathbb{R}$. This fact should be taken into account when studying the algebraic properties of $\mathbb{R}^+$ w. r. t. addition.

Let us mention that c-addition plays an important role in real analysis. In particular, this operation appears whenever the triangle inequality $|a + b| \leq |a| + |b|$ is used. Indeed, in the nontrivial case when a, b are of different signs the triangle inequality obtains the form $||a| - |b|| \leq |a| + |b|$, that is $|a| +^- |b| \leq |a| + |b|$. In the case of e-numbers the latter reads as $A +^- B \leq A + B$.

In the sequel we focus our attention on the operation c-addition (1). The operation c-addition "$+^-$" coincides with the so-called inner (or non-standard) addition (or inner subtraction) of symmetric intervals.

In the next section we review some of the properties of nonnegative real numbers relative to addition and multiplication by scalars focusing on the operation c-addition.

4 Properties of e-numbers relative to addition and order

4.1 E-numbers: addition and order

We first review the algebraic properties of the system of e-numbers $(\mathbb{R}^+, +, \leq)$ in comparison with the properties of $(\mathbb{R}, +, \leq)$ as reviewed in subsection 3.1.

Properties i)–iii) are satisfied in $\mathbb{R}^+$. Property iv) fails as there is no additive inverse (opposite) in $(\mathbb{R}^+, +)$, so equation $A + X = 0$ has no solution when $A \neq 0$. Subtractability does not hold as well, as $A + X = B$ does not possess a solution in general. The cancellation property v) $A + X = B + X \Longrightarrow A = B$ and commutativity vi) $A + B = B + A$ hold true.

Order isotonicity also takes place in $\mathbb{R}^+$. Namely, for $A, B, C, D \in \mathbb{R}^+$ we have $A \leq B \iff A + C \leq B + C$, and $A \leq B,\ C \leq D \Longrightarrow A + C \leq B + D$.

Summarizing, we can say that $(\mathbb{R}^+, +, \leq)$ is an ordered cancellative commutative monoid. The monoid $(\mathbb{R}^+, +, \leq)$ possesses the following two properties:

P1. For $A, B \in \mathbb{R}^+$, $A \neq B$, exactly one of the equations $A + X = B$, $B + Y = A$ is solvable.

P2. For $A, B \in \mathbb{R}^+$ $A + B = 0$ implies $A = B = 0$.

Properties P1 and P2 permit us to correctly define operation c-addition by means of (1). We next consider some of the algebraic properties of c-addition, that is of the system $(\mathbb{R}^+, +^-, \leq)$.

4.2 The system $(\mathbb{R}^+, +^-, \leq)$

The following properties of c-addition "$+^-$" follow from Definition (1):

i) "$+^-$" is a closed (total) operation;

ii) "$+^-$" is "c-associative": $(A +^- B) +^- C = A +^- (B +^- C)$, if $B \geq A$ and $B \geq C$;

iii) $A +^- 0 = A$ for all $A \in \mathbb{R}^+$;

iv) there is an additive inverse; namely for all $A \in \mathbb{R}^+$ the element A is opposite to A itself, that is $A +^- A = 0$;

v) "c-cancellation law": $A +^- X = B +^- X \Longrightarrow A = B$ or $X + X = A + B$;

vi) "commutative law": $A +^- B = B +^- A$, for all $A, B \in \mathbb{R}^+$.

Remark. Property v) says that cancellation $A +^- X = B +^- X \Longrightarrow A = B$ holds true under the condition $X + X \neq A + B$. The case $X + X = A + B$ (or $X = 0.5(A + B)$ if multiplication by scalars is available) is clearly exceptional, which gives us the right to call this property a "conditional cancellation" (briefly: *c-cancellation*).

The next property links c-addition and the order relation.

Conditional inclusion isotonicity w. r. t. c-addition. Let $A, B, C, D \in \mathbb{R}^+$ be such that $A \geq B, C \leq D$. We have: if $A \leq C$, then $A +^- C \leq B +^- D$, if $B \geq D$, then $A +^- C \geq B +^- D$.

In the special case $D = C$ we obtain the following corollary. Let $A, B, C \in \mathbb{R}^+$ be such that $A \geq B$. We have: if $A \leq C$, then $A +^- C \leq B +^- C$, if $B \geq C$, then $A +^- C \geq B +^- C$.

4.3 The extended additive monoid $(\mathbb{R}^+, +, +^-, \leq)$

As mentioned, subtractability does not hold neither in $(\mathbb{R}^+, +)$, nor in $(\mathbb{R}^+, +^-)$. However, the operation c-addition "$+^-$" allows solving equations of the form $A + X = B$ in certain cases. Namely, using "$+^-$" we can solve equation $A + X = B$ when $A \leq B$ and we can solve equation $B + X = A$ when $A \geq B$. Thus c-addition plays a role in $(\mathbb{R}^+, +)$ analogous to the role of subtraction in the group $(\mathbb{R}, +)$.

We have shown that the algebraic system $(\mathbb{R}^+, +, \leq)$ possesses null and c-addition; thus the system can be fully denoted as $(\mathbb{R}^+, +, +^-, \leq)$ or as $(\mathbb{R}^+, +, 0, +^-, \leq)$. To emphasize that system $(\mathbb{R}^+, +, \leq)$ includes c-addition we shall call it *extended additive e-numbers system.*

Note that the solution of $A + X = B$ (when existing) can be expressed in terms of c-addition, and the solution of $A +^- X = B$ can be expressed in terms of usual addition. Thus, solutions of both $A + X = B$ and $A +^- X = B$ become possible under certain conditions. This property is called "conditional subtractability", briefly "c-subtractability". We formulate it as follows:

C-subtractability. i) For $A, B \in \mathbb{R}^+$, such that $A \leq B$, the unique solution of $A + X = B$ is $X = B +^- A$. ii) Equation $A +^- X = B$ has a solution $X = A + B$ for $A, B \in \mathbb{R}^+$. If $A, B \in \mathbb{R}^+$ are such that $A \geq B > 0$, then equation $A +^- X = B$ has one more solution $X = A +^- B$.

We shall next focus our attention on the algebraic properties of e-numbers with respect to both addition and multiplication by scalars.

5 The quasilinear e-numbers space

5.1 Addition and c-addition of e-vectors

The above considerations can be generalized component-wise for n-vectors, that is elements of the systems $(\mathbb{R}^n, +, \leq)$, resp. $(\mathbb{R}^{+n}, +, \leq)$, noticing that then the order relation "$\leq$" is not total (linear) but partial. Here $\mathbb{R}^n$ is the set of real vectors $a = (a_1, a_2, ..., a_n)$, and $\mathbb{R}^{+n}$ is the set of n-tuples $A = (A_1, A_2, ..., A_n)$, $A_i \geq 0$.

Component-wise generalizations of previous definitions such as $a = (A; \alpha) \in \mathbb{R}^n$ with $A = (A_1, A_2, ..., A_n) \in \mathbb{R}^{+n}, \alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \Lambda^n$, etc. are obvious. We define addition and c-addition in $\mathbb{R}^{+n}$ as follows.

Definition 2. For $A = (A_1, A_2, ..., A_n), B = (B_1, B_2, ..., B_n) \in \mathbb{R}^{+n}$, we define $A + B$ and $A +^- B$ by means of:

$$\begin{aligned} A + B &= (A_1, A_2, ..., A_n) + (B_1, B_2, ..., B_n) = (A_1 + B_1, ..., A_n + B_n), \\ A +^- B &= (A_1, A_2, ..., A_n) +^- (B_1, B_2, ..., B_n) = (A_1 +^- B_1, ..., A_n +^- B_n). \end{aligned}$$

Remark. Note that for $A, B \in \mathbb{R}^{+n}$, $n \geq 2$ the expression

$$A +^- B = \begin{cases} Y|_{B+Y=A} & \text{if } B \leq A; \\ X|_{A+X=B} & \text{if } A \leq B \end{cases}$$

does not describe $A +^- B$ in the case when neither $A \leq B$ nor $B \leq A$ hold in $\mathbb{R}^{+n}$.

Let $a = (a_1, a_2, ..., a_n), b = (b_1, b_2, ..., b_n) \in \mathbb{R}^n$ be two real vectors presented in sm-form, that is $a_i = (A_i; \alpha_i)$, $b_i = (B_i; \beta_i)$, $i = 1, \ldots, n$. From (2) we have:

$$a_i + b_i = (A_i; \alpha_i) + (B_i; \beta_i) = (A_i +^{\alpha_i \beta_i} B_i;\ \mu(a_i, b_i)), \qquad (3)$$

wherein

$$\mu(a_i, b_i) = \mu((A_i; \alpha_i), (B_i; \beta_i)) = \begin{cases} \alpha_i & \text{if } B_i \leq A_i, \\ \beta_i & \text{if } B_i > A_i. \end{cases}$$

Hence, to be able to perform addition of real numbers in sm-form we need the following operation between two vectors of nonnegative components[1] $A = (A_1, A_2, ..., A_n), B = (B_1, B_2, ..., B_n) \in \mathbb{R}^{+n}$:

$$\begin{aligned} A +^{\lambda} B &= (A_1, A_2, ..., A_n) +^{\lambda} (B_1, B_2, ..., B_n) \\ &= (A_1 +^{\lambda_1} B_1, A_2 +^{\lambda_2} B_2, ..., A_n +^{\lambda_n} B_n), \end{aligned} \tag{4}$$

where $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \Lambda^n$ is a boolean vector (n-tuple) of signs $\pm$.

Definition (4) generalizes the definitions of $A+B$ and $A+^{-}B$ given in Definition 2.

Using the general definition (4) of $A +^{\lambda} B$ we can write down the sum of two real vectors in sm-form briefly as follows:

$$a + b = (A; \alpha) + (B; \beta) = (A +^{\alpha\beta} B;\ \mu(a, b)), \tag{5}$$

wherein $A, B \in \mathbb{R}^{+n}$, $\alpha, \beta, \mu \in \Lambda^n$. By $\alpha\beta$ we mean the sign vector

$$\alpha\beta = (\alpha_1, \alpha_2, ..., \alpha_n)(\beta_1, \beta_2, ..., \beta_n) = (\alpha_1\beta_1, \alpha_2\beta_2, ..., \alpha_n\beta_n).$$

5.2 Multiplication by scalars

We now focus our attention to multiplication by scalars. Introducing in $\mathbb{R}^n$ multiplication by scalars from the real ordered field $\mathbb{R} = (\mathbb{R}, +, \cdot, \leq)$, we arrive to the familiar vector space $(\mathbb{R}^n, +, \mathbb{R}, \cdot, \leq)$. Multiplication of a real vector $a = (A; \alpha) \in \mathbb{R}^n$ in sm-form by a scalar $c \in \mathbb{R}$ is given by

$$c \cdot (A; \alpha) = (|c| \cdot A; \sigma(c)\alpha). \tag{6}$$

In (6) $\sigma(c)$ is the sign of the scalar c, resp. $\sigma(c)\alpha$ is equal to either α or $-\alpha$ depending on the sign of c. Relation (6) shows that multiplication of a real vector by scalars induces a new "quasivector" multiplication

[1] We should be careful with using the term vector for an n-tuple of e-numbers as such n-tupples are not elements of a vector space

by real scalars "$*$" in the "e-numbers space" $(\mathbb{R}^{+n}, +, \mathbb{R}, *, \leq)$ to be defined as follows

Definition 3. Quasivector multiplication by real scalars "$*$" is defined as

$$c * A = |c| \cdot A,\ c \in \mathbb{R},\ A \in \mathbb{R}^{+n}. \tag{7}$$

Componentwise, (7) reads:

$$\begin{aligned} c * A &= |c| \cdot A = |c| \cdot (A_1, A_2, ..., A_n) \\ &= (|c|A_1, |c|A_2, ..., |c|A_n). \end{aligned}$$

Using quasivector multiplication by scalars "$*$" relation (6) becomes

$$c \cdot (A; \alpha) = (c * A; \sigma(c)\alpha),\ c \in \mathbb{R},\ A \in \mathbb{R}^{+n}. \tag{8}$$

The quasivector multiplication by scalars "$*$" possesses the following properties.

Proposition 1. For $A, B \in \mathbb{R}^{+n}$, all $s, t \in \mathbb{R}$ and $\lambda \in \Lambda^n$:

$$\begin{aligned} s * (t * A) &= (st) * A, & (9)\\ 1 * A &= A, & (10)\\ s * (A +^{\lambda} B) &= s * A +^{\lambda} s * B, & (11)\\ (s + t) * A &= s * A +^{\sigma(s)\sigma(t)} t * A, & (12)\\ A \leq B &\Longrightarrow \gamma * A \leq \gamma * B, & (13)\\ (-1) * A &= A, & (14) \end{aligned}$$

Proof. Properties (9), (10), (13) and (14) follow trivially from Definition (7). To prove relations (11–12) we start from analogous relations for real vectors $a, b \in \mathbb{R}^n$ written in sm-form: $a = (A; \alpha), b = (B; \beta)$, namely the familiar distributive relations $s(a + b) = sa + sb$ and $(s + t)a = sa + ta$, $s, t \in \mathbb{R}$.

We first prove (11). To this end we write consecutively:

$$\begin{aligned} s(a+b) &= sa+sb; \\ s((A;\alpha)+(B;\beta)) &= s(A;\alpha)+s(B;\beta); \\ s(A+^{\alpha\beta} B;\mu(a,b)) &= (s*A;\ \sigma(s)\alpha)+(s*B;\ \sigma(s)\beta); \\ (s*(A+^{\alpha\beta} B);\ \sigma(s)\mu(a,b)) &= (s*A+^{\alpha\beta} s*B;\ \mu'); \\ s*(A+^{\alpha\beta} B) &= s*A+^{\alpha\beta} s*B, \end{aligned}$$

obtaining finally (11): $s*(A+^{\lambda} B = s*A+^{\lambda} s*B$.

To prove (12) we write consecutively:

$$\begin{aligned} (s+t)(A;\alpha) &= s(A;\alpha)+t(A;\alpha); \\ ((s+t)*A;\ \sigma(s+t)\alpha) &= (s*A;\sigma(s)\alpha)+(t*A;\sigma(t)\alpha) \\ &= (s*A+^{\sigma(s)\sigma(t)} t*A;\ \mu(sa,ta)), \end{aligned}$$

giving the needed (12): $(s+t)*A = s*A+^{\sigma(s)\sigma(t)} t*A$. □

Remark. The first five properties (9–13) are characteristic for a general quasilinear space such as the space of intervals and the space of convex bodies. The last one is characteristic for symmetric quasilinear spaces such as the e-numbers(equivalently: symmetric intervals) and symmetric convex bodies. Note also that relation (12) reads:

$$(s+t)*A = \begin{cases} s*A+t*A & \text{if} \quad st \geq 0; \\ s*A+^{-} t*A & \text{if} \quad st < 0, \end{cases}$$

showing that the familiar second distributive law holds under the restriction $st \geq 0$. Relation (12) is called "quasidistributive law". Sometimes the quasidistributive law is postulated in the form $s*A+t*A$ if $st \geq 0$; then the remaining part $s*A+^{-} t*A$ if $st < 0$, is derived as a logical consequence.

5.3 Linear combinations and linear dependency

Recall that k real vectors $c_1, c_2, ..., c_k \in \mathbb{R}^n$ are linearly dependent if there exist k real numbers $\alpha_1, \alpha_2, ..., \alpha_k \in \mathbb{R}$, not all equal to zero, such that

$$\sum_{i=1}^{k} \alpha_i c_i = \alpha_1 c_1 + \alpha_2 c_2 + ... + \alpha_k c_k = 0.$$

W.l.g. we shall assume that $\alpha_1 \geq 0$.

Our next aim is to suitably modify this definition for e-vectors. To this end let us represent in sm-form the linear combination of k real vectors that appears in the above definition, namely:

$$c = \sum_{i=1}^{k} \alpha_i c_i = \alpha_1 c_1 + \alpha_2 c_2 + ... + \alpha_k c_k, \tag{15}$$

wherein $c_i = ({c_i}^{(1)}, {c_i}^{(2)}, ..., {c_i}^{(n)}) \in \mathbb{R}^n,\ \alpha_i \in \mathbb{R},\ i = 1, ..., k$. We substitute each component of c_i by its sm-form: $c_i = (C_i; \gamma_i)$, resp. ${c_i}^{(j)} = ({C_i}^{(j)}; {\gamma_i}^{(j)}),\ j = 1, ..., n,\ i = 1, ..., k$. As we are interested in the linear combination of nonnegative vectors, we assume that ${\gamma_i}^{(j)} = +,\ j = 1, ..., n,\ i = 1, ..., k$, so that ${c_i}^{(j)} = ({C_i}^{(j)}; +),\ j = 1, ..., n,\ i = 1, ..., k$. In vector notation the latter reads: $c_i = (C_i; +),\ i = 1, ..., k$.

For simplicity we start with $k = 2$, that is with a linear combination involving two real vectors $c_i \in \mathbb{R}^n,\ i = 1, 2$. Using the formulae for addition of two vectors in sm-form (3), (4) and for multiplication by scalars (6) we obtain in vector notation (5):

$$\begin{aligned} c = \alpha_1 c_1 + \alpha_2 c_2 &= \alpha_1 (C_1;\ +) + \alpha_2 (C_2;\ +) \\ &= (\alpha_1 * C_1;\ \sigma(\alpha_1)) + (\alpha_2 * C_2;\ \sigma(\alpha_2)) \\ &= (\alpha_1 * C_1 +^{\lambda_2} \alpha_2 * C_2;\ \mu_2), \end{aligned} \tag{16}$$

where $\lambda_2 = \sigma(\alpha_1)\sigma(\alpha_2) \in \Lambda$, $\mu_2 = \mu(\alpha_1 c_1, \alpha_2 c_2) \in \Lambda^n$.

The above equality is written component-wise as follows:

$$\begin{aligned} \alpha_1 c_1^{(j)} + \alpha_2 c_2^{(j)} &= (\alpha_1 * {C_1}^{(j)};\ \sigma(\alpha_1)) + (\alpha_2 * {C_2}^{(j)};\ \sigma(\alpha_2)) \\ &= (\alpha_1 * {C_1}^{(j)} +^{\sigma(\alpha_1)\sigma(\alpha_2)} \alpha_2 * {C_2}^{(j)};\ {\mu_2}^{(j)}),\ j = 1, ..., n, \end{aligned} \tag{17}$$

where ${\mu_2}^{(j)}$ is given by

$$ {\mu_2}^{(j)} = \begin{cases} \sigma(\alpha_1) & \text{if } \ \alpha_1 * {C_1}^{(j)} \geq \alpha_2 * {C_2}^{(j)}; \\ \sigma(\alpha_2) & \text{otherwise.} \end{cases} $$

With respect to the sign of the linear combination $c = (C; \mu_2)$ some of its component may be negative as we have assumed $\sigma(\alpha_1) = +$, but α_2 may be negative and $\sigma(\alpha_2) = -$. However, w. l. g. we shall consider only linear combinations which are nonnegative, $c = \alpha_1 c_1 + \alpha_2 c_2 \geq 0$, so that $c = (C; +)$ and therefore $\mu_2 = +$ is a constant sign vector.

We define "linear dependence" for two vectors $C_1, C_2 \in {\mathbb{R}^+}^n$ as follows.

Definition. Two vectors $C_1, C_2 \in {\mathbb{R}^+}^n$ are "linearly dependent" if there exists a nonzero pair $(\alpha_1, \alpha_2) \neq 0$, $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1 \geq 0$, such that

$$ \alpha_1 * C_1 +^{\sigma(\alpha_2)} \alpha_2 * C_2 = 0. $$

We next proceed similarly with a linear combination of three real e-vectors $c_1, c_2, c_3 \in \mathbb{R}^n$. Assuming again that all c_i are nonnegative and $\sigma(\alpha_1) = +$, $\sigma(\mu_2) = +$, we have:

$$ \begin{aligned} c = \alpha_1 c_1 + \alpha_2 c_2 + \alpha_3 c_3 &= (\alpha_1 * C_1;\ \sigma(\alpha_1)) + (\alpha_2 * C_2;\ \sigma(\alpha_2)) + (\alpha_3 * C_3;\ \sigma(\alpha_3)) \\ &= (\alpha_1 * C_1 +^{\sigma(\alpha_1)\sigma(\alpha_2)} \alpha_2 * C_2; \mu_2) + (\alpha_3 * C_3;\ \sigma(\alpha_3)) \\ &= ((\alpha_1 * C_1 +^{\sigma(\alpha_2)} \alpha_2 * C_2) +^{\sigma(\alpha_3)} \alpha_3 * C_3;\ \mu_3), \end{aligned} $$

wherein the sign vector μ_3 has the following components:

$$ {\mu_3}^{(j)} = \begin{cases} {\mu_2}^{(j)} & \text{if } \ (\alpha_1 * C_1 +^{\sigma(\alpha_2)} \alpha_2 * C_2)^{(j)} \geq \alpha_3 * {C_3}^{(j)}; \\ \sigma(\alpha_3) & \text{otherwise.} \end{cases} $$

The sign μ_3 of c may not be positive in general. However, as in the case $k = 2$, we again assume w. l. g. that c is positive and thus $\mu_3 = 0$, that is $c = (C; +)$.

The above calculations suggest that we can define "linear dependence" for three vectors $C_1, C_2, C_3 \in {\mathbb{R}^+}^n$ as follows:

Definition A. The e-vectors $C_1, C_2, C_3 \in \mathbb{R}^{+n}$ are "linearly dependent" if there exists a nonzero triple $(\alpha_1, \alpha_2, \alpha_3) \neq 0$, $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, $\alpha_1 \geq 0$, such that

$$(\alpha_1 * C_1 +^{\sigma(\alpha_2)} \alpha_2 * C_2) +^{\sigma(\alpha_3)} \alpha_3 * C_3 = 0.$$

Since the coefficients α_2, α_3 are arbitrary reals, their signs are also arbitrary, so that the above Definition A is equivalent to the following one:

Definition B. The e-vectors $C_1, C_2, C_3 \in \mathbb{R}^{+n}$ are "linearly dependent" if there exists a nonzero triple $(\alpha_1, \alpha_2, \alpha_3) \neq 0$, $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, and signs $\lambda_i \in \Lambda$, $i = 1, 2$, such that

$$(\alpha_1 * C_1 +^{\lambda_1} \alpha_2 * C_2) +^{\lambda_2} \alpha_3 * C_3 = 0. \tag{18}$$

Definition B is generalized for arbitrary number k of e-vectors as follows:

Definition C. The e-vectors $C_1, C_2, ..., C_k \in \mathbb{R}^{+n}$ are "linearly dependent" if there exists a nonzero vector $(\alpha_1, \alpha_2, ..., \alpha_k) \neq 0$, $\alpha_1, \alpha_2, ..., \alpha_k \in \mathbb{R}$, and signs $\lambda_i \in \Lambda$, $i = 1, 2, ..., k-1$, such that

$$\alpha_1 * C_1 +^{\lambda_1} \alpha_2 * C_2 +^{\lambda_2} ... +^{\lambda_{k-1}} \alpha_k * C_k = 0, \tag{19}$$

with order of executions of the operations "$+^{\lambda_i}$" in (19) from left to right.

Definition C can be generalized for interval vectors. Indeed, we can think of the e-vectors $C_1, C_2, ..., C_k \in \mathbb{R}^{+n}$ as of symmetric interval vectors (having midpoints zero). Now let us think of C_k as of intervals with arbitrary midpoints, that is $C_1, C_2, ..., C_k \in \mathbf{I}\mathbb{R}^n$, where $C_k = (C_k'; C_k'')$ with $C_1', C_2', ..., C_k' \in \mathbb{R}^n$, $C_1'', C_2'', ..., C_k'' \in \mathbb{R}^{+n}$. Then we have:

Definition D. The intrval vectors $C_1, C_2, ..., C_k \in \mathbf{I}\mathbb{R}^n$ are "linearly dependent" if there exists a nonzero vector $(\alpha_1, \alpha_2, ..., \alpha_k) \neq 0$, $\alpha_1, \alpha_2, ..., \alpha_k \in \mathbb{R}$, and signs $\lambda_i \in \Lambda$, $i = 1, 2, ..., k-1$, such that

$$\alpha_1 * C_1 +^{\lambda_1} \alpha_2 * C_2 +^{\lambda_2} ... +^{\lambda_{k-1}} \alpha_k * C_k = 0, \tag{20}$$

with order of executions of the operations "$+^{\lambda_i}$" in (20) from left to right.

Note that the condition (20) reduces to two separate conditions, one for the midpoints (which is the well-known condition for real vectors) and one for the radii, which is condition (19) for e-numbers.

6 Conclusions

In the present work we show that:

i) addition of real numbers naturally induces the operation c-addition of non-negative numbers (distance, modulus of the difference);

ii) the operation c-addition of non-negative numbers enriches the additive monoidal system of non-negative numbers up to a structure close to a group where many typically group operations can be performed under certain conditions;

iii) the operation c-addition of non-negative numbers is fundamental in real analysis, in interval analysis, and resp. in error analysis;

iv) the introduction of the operation "multiplication (of nonnegative numbers) by (real) scalars" leads to a special algebraic structure "quasilinear space", close but yet different from linear spaces;

v) using the c-associative property one naturally arrives to the concept of linear independency of e-vectors, resp. interval vectors.

Our approach in this work is based on the "approximate-number"-concept. An attempt to define linear independence of interval vectors based on the "set-interval"-concept has been made in [1]. Our approach based on the "approximate number"-concept proves to be more natural and methodologically simpler; it leads to simpler definitions and expressions. To study comparatively the two approaches is a future task.

Acknowledgements. This work was prepared while the author visited the Department of Mathematics and Applied Mathematics at the University of Pretoria in September 2011. The author is grateful to the staff of the Department for their interest during the seminar lecture and their hospitality.

References

[1] Ahn, Hyo-Sung, Kevin L. Moore, YangQuan Chen, Linear Independency of Interval Vectors and Its Applications to Robust Controllability Tests, Proc. 44th IEEE Conf. on Decision and Control, and the European Control Conf. 2005, Seville, Spain, Dec. 12–15, 2005, 8070–8075.
`http://ieeexplore.ieee.org/xpls/abs_all.jsp?arnumber=1583468`

[2] Edmonson, W., G. Melquiond. IEEE Interval Standard Working Group - P1788: Current Status, Proc. 19th IEEE Symposium on Computer Arithmetic, 2009. ARITH 2009, 231–234.
`http://grouper.ieee.org/groups/1788/`

[3] Hayes, B., A Lucid Interval, American Scientist 91(6), 484–488, 2003. downloadable from: `http://bit-player.org/pubs`

[4] Markov, S., Extended interval arithmetic. *Compt. rend. Acad. bulg. Sci.* **30**, 9, 1977, 1239–1242.

[5] Markov, S., Calculus for interval functions of a real variable. *Computing* **22**, 325–337 (1979).

[6] Markov, S., On the Presentation of Ranges of Monotone Functions Using Interval Arithmetic, Interval Computations 4 (6), 1992, 19–31.

[7] Markov, S., and K. Okumura. The contribution of T. Sunaga to interval analysis and reliable computing. Developments in Reliable Computing, 1999.

[8] Markov, S., On Quasilinear Spaces of Convex Bodies and Intervals, Journal of Computational and Applied Mathematics 162 (1), 93–112, 2004.

[9] Markov, S., N. Hayes, On the arithmetic of errors, *Serdica J. Computing* 4 (4), 447–462, 2010.

[10] Sunaga, T., Theory of interval algebra and its application to numerical analysis, 1958.

[11] http://en.wikipedia.org/wiki/Interval_arithmetic

[12] http://interval.louisiana.edu/reliable-computing-journal/

[13] http://interval.louisiana.edu/mailing_list_form.html

[14] http://www.cs.utep.edu/interval-comp/