Article Biomath Communications 6 (2019)

Biomath Communications

www.biomathforum.org/biomath/index.php/conference

A Study on the Unit–logistic, Unit–Weibull and Topp–Leone Cumulative Sigmoids

Anton Iliev[1], Asen Rahnev[2],
Nikolay Kyurkchiev[3] and Svetoslav Markov[4]

[1,2,3]Faculty of Mathematics and Informatics,
University of Plovdiv Paisii Hilendarski,
24, Tzar Asen Str., 4000 Plovdiv, Bulgaria,
e-mails: aii@uni-plovdiv.bg, assen@uni-plovdiv.bg,
nkyurk@uni-plovdiv.bg,

[4]Institute of Mathematics and Informatics,
Bulgarian Academy of Sciences,
Acad. G. Bonchev Str., Bl. 8, 1113 Sofia, Bulgaria,
e-mail: smarkov@bio.bas.bg

Abstract. In this paper we study the one–sided Hausdorff approximation of the Heaviside step function by a families of Unit–Logistic (UL), Unit–Weibull (UW) and Topp–Leone (TL) cumulative sigmoids. The estimates of the value of the best Hausdorff approximation obtained in this article can be used in practice as one possible additional criterion in "saturation" study. Numerical examples are presented using *CAS MATHEMATICA*.

AMS Subject Classification: 41A46

Citation: Anton Iliev, Asen Rahnev, Nikolay Kyurkchiev, Svetoslav Markov, A Study on the Unit–logistic, Unit–Weibull and Topp–Leone Cumulative Sigmoids, Biomath Communications 6, pp. 1-16,
`https://doi.org/10.11145/bmc.2019.03.167`

Keywords: Unit–Logistic (UL) cumulative sigmoid, Unit–Weibull (UW) cumulative sigmoid, Topp–Leone (TL) cumulative sigmoids, Hausdorff distance, upper and lower bounds

1 Introduction

In [1] the authors introduced a new probability distribution with support on $(0,1)$ and named the distribution as Unit–Logistic Distribution (ULD).

In [2] the authors introduced an alternative parametrization, where one parameter is the median.

The corresponding cumulative distribution function is written as [3]:

$$M(t) = \left(1 + \left(\frac{\mu(1-t)}{t(1-\mu)}\right)^{\beta}\right)^{-1}, \tag{1}$$

where $0 \le t \le 1$ and $0 \le \mu \le 1$ is the median.

In [4] the authors introduced a new probability distribution with support on $(0,1)$ and named the distribution as Unit–Weibull Distribution (UWD). The corresponding cumulative distribution function is written as:

$$M_1(t) = e^{-\alpha(-\ln t)^{\beta}}, \tag{2}$$

where $0 \le t \le 1$ and $\alpha,\ \beta > 0$.

The Topp–Leone (TL) distribution was originally proposed by Topp and Leone (1955) [17] as an alternative to beta distribution and it has been applied for some failure data. The corresponding cumulative distribution function is written as:

$$M_2(t) = t^{\alpha}(2-t)^{\alpha}, \tag{3}$$

where $0 \le t \le 1$ and $\alpha > 0$.

Such cumulative probability distributions can be used with success in approximating parameterized data in the field of "virus-theory", insurance mathematic and population dynamics.

2 Main Results

In this Section we study the Hausdorff approximation [5] of the Heaviside step function by families of the Unit–Logistic (UL), Unit–Weibull (UW) and Topp–Leone (TL) cumulative sigmoids.

2.1 A note on the Unit–Logistic cumulative sigmoid

Evidently $M(t_0 = \mu) = \frac{1}{2}$. The Hausdorff distance d between the function

$$h_\mu(t) = \begin{cases} 0, & \text{if} \quad t < \mu, \\ [0,1], & \text{if} \quad t = \mu, \\ 1, & \text{if} \quad t > \mu, \end{cases}$$

and the sigmoid (1) satisfies the relation

$$M(t_0 + d) = 1 - d. \tag{4}$$

The following theorem gives upper and lower bounds for d.

Theorem 1. Let

$$\begin{aligned} &p = -\tfrac{1}{2}, \\ &q = 1 + \tfrac{\beta}{4\mu(1-\mu)}, \\ &r = 2.1q. \end{aligned} \tag{5}$$

For the Hausdorff distance d between $h_\mu(t)$ and the sigmoid (1) the following inequalities hold for $q > \dfrac{e^{1.05}}{2.1}$:

$$d_l = \frac{1}{r} < d < \frac{\ln r}{r} = d_r. \tag{6}$$

Proof. Let us examine the function:

$$F(d) = M(t_0 + d) - 1 + d. \tag{7}$$

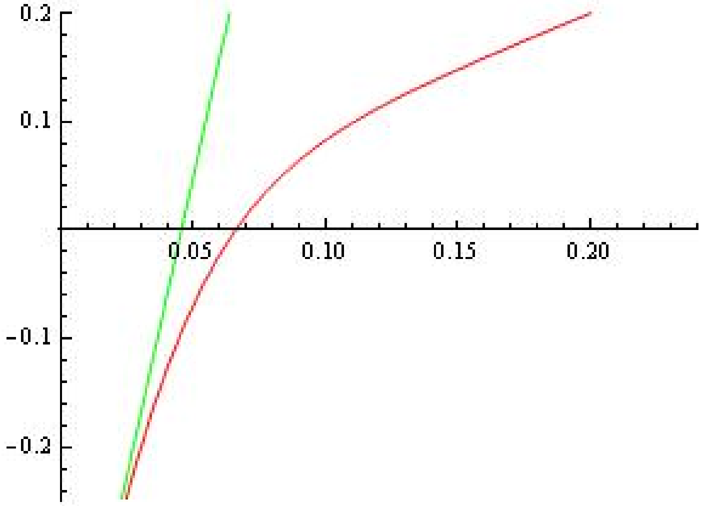

Figure 1: The functions $F(d)$ and $G(d)$ for $\mu = 0.5$; $\beta = 10$.

From $F'(d) > 0$ we conclude that function F is increasing.

Consider the function

$$G(d) = p + qd. \tag{8}$$

From Taylor expansion we obtain $G(d) - F(d) = O(d^2)$. Hence $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see Fig. 1).

In addition $G'(d) > 0$. Further, for $q > \dfrac{e^{1.05}}{2.1}$ we have $G(d_l) < 0$ and $G(d_r) > 0$.

This completes the proof of the theorem.

2.2 Numerical examples

The model (1) for $\mu = 0.5$; $\beta = 2$ is visualized on Fig. 2. From the nonlinear equation (4) and inequalities (6) we have: $d = 0.180552$, $d_l = 0.15873$, $d_r = 0.292151$.

The model (1) for $\mu = 0.5$; $\beta = 10$ is visualized on Fig. 3. From the nonlinear equation (4) and inequalities (6) we have: $d = 0.0659004$, $d_l = 0.04329$, $d_r = 0.135923$.

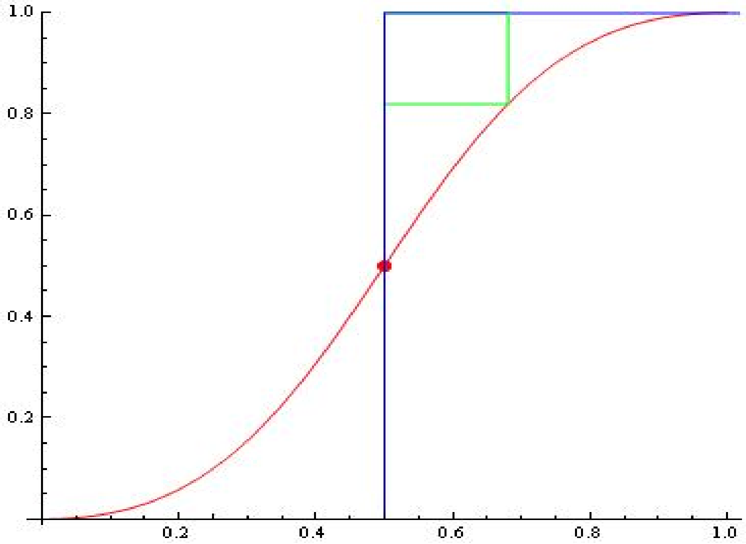

Figure 2: The model (1) for $\mu = 0.5$; $\beta = 2$; H–distance $d = 0.180552$, $d_l = 0.15873$, $d_r = 0.292151$.

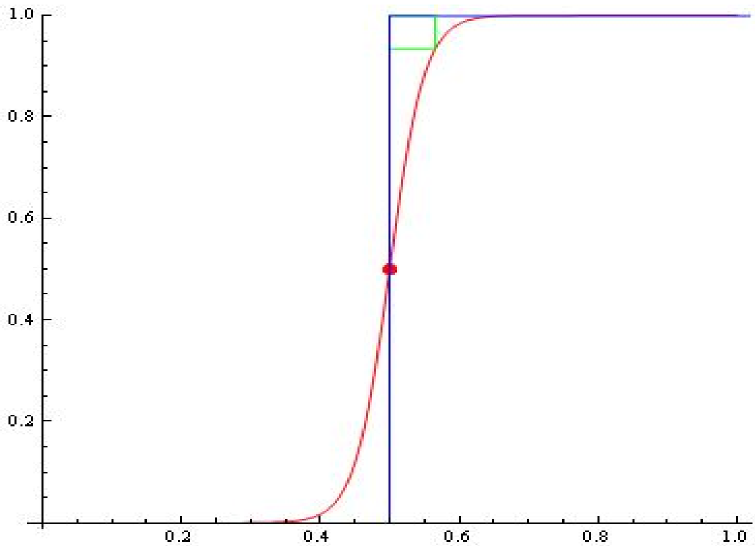

Figure 3: The model (1) for $\mu = 0.5$; $\beta = 10$; H–distance $d = 0.0659004$, $d_l = 0.04329$, $d_r = 0.135923$.

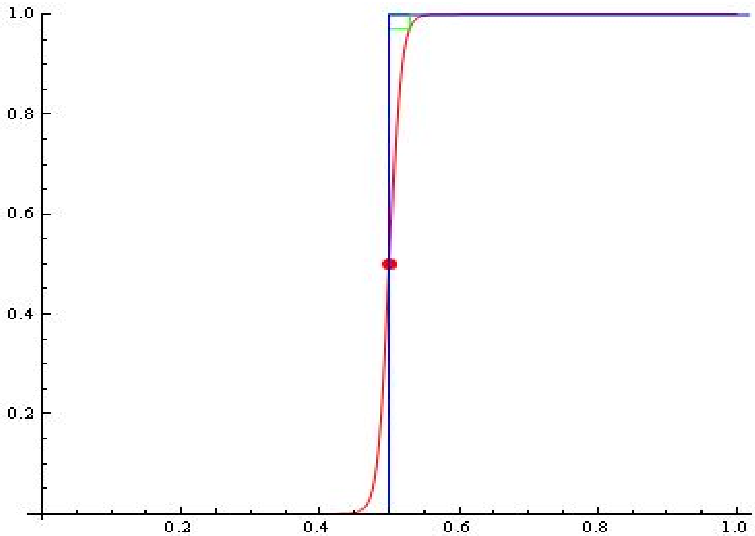

Figure 4: The model (1) for $\mu = 0.5$; $\beta = 30$; H–distance $d = 0.0291741$, $d_l = 0.015361$, $d_r = 0.0641463$.

The model (1) for $\mu = 0.5$; $\beta = 30$ is visualized on Fig. 4. From the nonlinear equation (5) and inequalities (6) we have: $d = 0.0291741$, $d_l = 0.015361$, $d_r = 0.0641463$.

From the graphics it can be seen that the "saturation" is faster.

Some computational examples using relations (3) are presented in Table 1. The last column of Table 1 contains the values of d computed by solving the nonlinear equation (3). From Table 1, it can be seen that the right estimates for the value of the best Hausdorff distance are precise.

We examine the experimental (parameterized) data (Biomass for Xantobacter autotrophycum with electric field). The appropriate fitting of the data by the model (1) with $\mu = 0.5$ and $\beta = 2.0854$ is visualized on Fig. 5.

β	d_l	d_r	d from (4)
2	0.15873	0.292151	0.180552
10	0.04329	0.135923	0.0659004
20	0.0226757	0.0858608	0.0397299
30	0.015361	0.0641463	0.0291741
40	0.0116144	0.0517481	0.0233246
50	0.00933707	0.0436392	0.019562

Table 1: Bounds for d computed by (6) for fixed $\mu = 0.5$ and some values of scale parameter β.

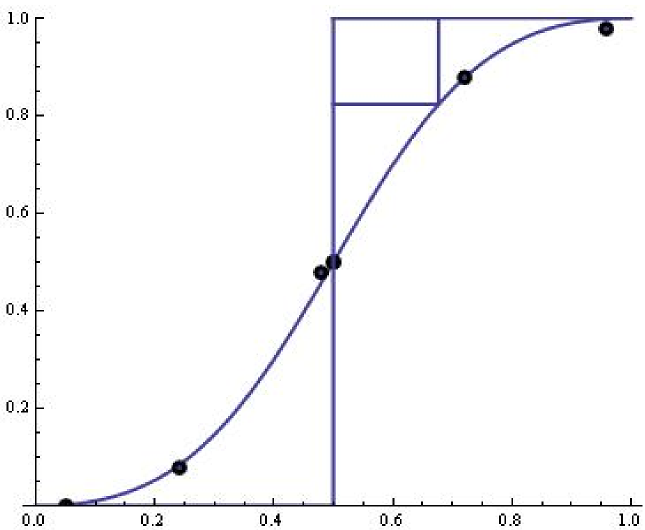

Figure 5: The model (1) for $\mu = 0.5$; $\beta = 2.0854$; The Hausdorff distance: $d = 0.176608$.

2.3 A note on the Unit–Weibull cumulative sigmoid

We study the Hausdorff approximation of the *shifted Heaviside step function* $h_{t_0}(t)$ by the family (2).

Let

$$t_0 = e^{-\left(\frac{\ln 2}{\alpha}\right)^{\frac{1}{\beta}}}. \tag{9}$$

Evidently, $M_1(t_0) = \frac{1}{2}$.

The one–sided Hausdorff distance d between the function $h_{t_0}(t)$ and the sigmoid (2) satisfies the relation

$$M_1(t_0 + d) = 1 - d. \tag{10}$$

The following theorem gives upper and lower bounds for d.

Theorem 2. Let

$$\begin{aligned} p &= -\tfrac{1}{2}, \\ q &= 1 + \tfrac{\alpha\beta}{2} \frac{\left(\frac{\ln 2}{\alpha}\right)^{\frac{\beta-1}{\beta}}}{e^{-\left(\frac{\ln 2}{\alpha}\right)^{\frac{1}{\beta}}}} \\ r &= 2.1q. \end{aligned} \tag{11}$$

For the one–sided Hausdorff distance d between $h_{t_0}(t)$ and the sigmoid (2) the following inequalities hold for $q > \dfrac{e^{1.05}}{2.1}$:

$$d_l = \frac{1}{r} < d < \frac{\ln r}{r} = d_r. \tag{12}$$

Proof. Let us examine the function:

$$F(d) = M(t_0 + d) - 1 + d. \tag{13}$$

and

$$G(d) = p + qd. \tag{14}$$

The proof follows the ideas given in this paper.

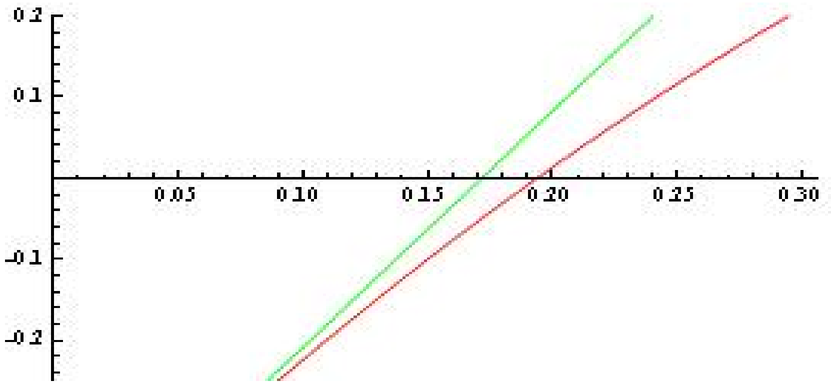

Figure 6: The functions $F(d)$ and $G(d)$ for $\alpha = 1$; $\beta = 2$.

We note that from Taylor expansion we obtain

$$G(d) - F(d) = O(d^2),$$

i.e. $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see Fig. 6).

Further, for $q > \dfrac{e^{1.05}}{2.1}$ we have $G(d_l) < 0$ and $G(d_r) > 0$.

This completes the proof of the theorem.

The model (2) for $\alpha = 1$; $\beta = 2$ is visualized on Fig. 7. From the nonlinear equation (10) and inequalities (12) we have: $d = 0.193768$, $d_l = 0.163404$, $d_r = 0.296011$.

2.4 A note on the Topp–Leone cumulative sigmoid

We study the Hausdorff approximation of the *shifted Heaviside step function* $h_{t_0}(t)$ by the family (3).

Let $t_0 \in (0, 1)$ is the solution of the nonlinear equation

$$t_0(2 - t_0) - 0.5^{\frac{1}{\alpha}} = 0 \qquad (15)$$

i.e. $M_2(t_0) = \frac{1}{2}$.

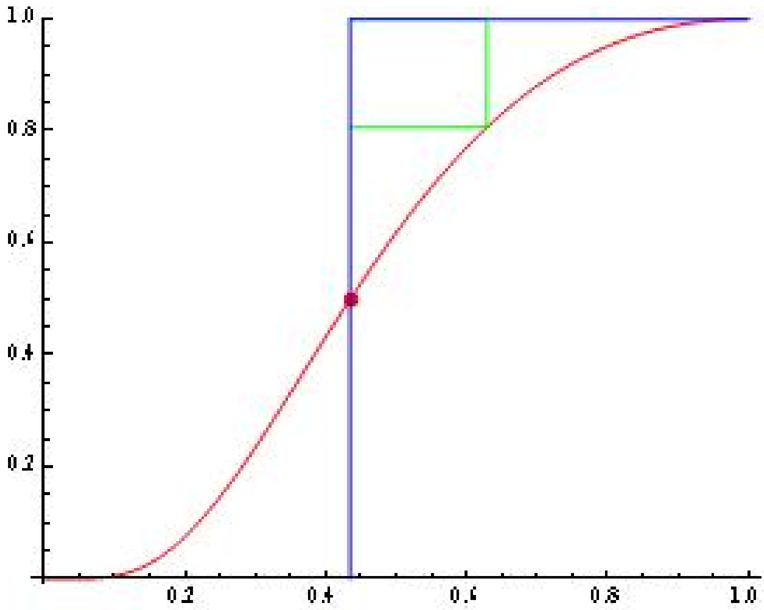

Figure 7: The model (2) for $\alpha = 1$; $\beta = 2$; $t_0 = 0.434937$; H-distance $d = 0.193768$, $d_l = 0.163404$, $d_r = 0.296011$.

The one–sided Hausdorff distance d between the function $h_{t_0}(t)$ and the sigmoid (3) satisfies the relation

$$M_2(t_0 + d) = 1 - d. \tag{16}$$

The following theorem gives upper and lower bounds for d.

Theorem 3. Let

$$\begin{aligned} p &= -\tfrac{1}{2}, \\ q &= 1 + 2\alpha \left(\tfrac{1}{2}\right)^{\frac{\alpha-1}{\alpha}} (1 - t_0) \\ r &= 2.1q. \end{aligned} \tag{17}$$

For the one–sided Hausdorff distance d between $h_{t_0}(t)$ and the sigmoid

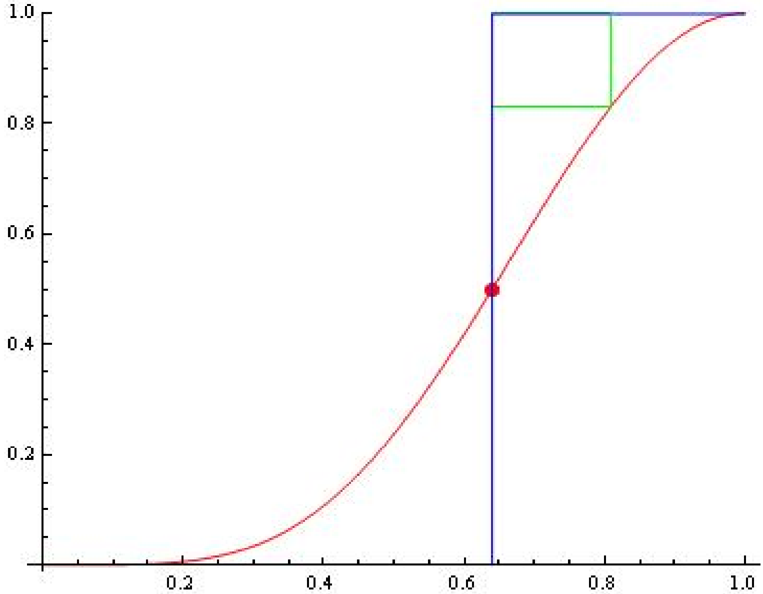

Figure 8: The model (3) for $\alpha = 5$; $t_0 = 0.640209$; H-distance $d = 0.169087$, $d_l = 0.15529$, $d_r = 0.289222$.

(3) the following inequalities hold for $q > \dfrac{e^{1.05}}{2.1}$:

$$d_l = \frac{1}{r} < d < \frac{\ln r}{r} = d_r. \tag{18}$$

The proof follows the ideas given in this paper and will be omitted.

The model (3) for $\alpha = 5$ is visualized on Fig. 8. From the nonlinear equation (16) and inequalities (18) we have: $d = 0.169087$, $d_l = 0.15529$, $d_r = 0.289222$.

For some approximation, computational and modelling aspects, see [6]–[13], [18]–[27].

The results obtained in this paper can be used when controlling growth in Software Reliability Models, see [15]–[16].

Acknowledgment

This paper is supported by the National Scientific Program "Information and Communication Technologies for a Single Digital Market in Science, Education and Security (ICTinSES)", financed by the Ministry of Education and Science.

References

[1] P. R. Tadikamalla, N. L. Johnson, Systems of frequency curves generated by transformations of Logistic variables, Biometrika, **69**(2) 1982, 461–465.

[2] R. F. da Paz, N. Balakrishnan, J. L. Bazan, L–logistic distribution: Properties, inference and an application to study poverty and inequality in Brazil, Tech. Rep., Programa Interinstitucional de Poa Graduacao em Estitistics UFSCar – USP, Sao Carles, SP, Brazil, 2016.

[3] A. Menezes, J. Mazucheli, S. Dey, The Unit–logistic distribution: different methods of estimation, Pesquisa Operacional, **38**(3) 2018, 555–578.

[4] A. Menezes, J. Mazucheli, M. Ghitany, The Unit–Weibull distribution and associated inference, Journal of Applied and Prob. Stat., **13**(2) 2018, 1–22.

[5] B. Sendov, Hausdorff Approximations, Kluwer, Boston, 1990.

[6] N. Kyurkchiev, S. Markov, On the Hausdorff distance between the Heaviside step function and Verhulst logistic function, J. Math. Chem., **54**(1) 2016, 109–119.

[7] N. Kyurkchiev, S. Markov, Sigmoid functions: Some Approximation and Modelling Aspects, LAP LAMBERT Academic Publishing, Saarbrucken, 2015, ISBN: 978-3-659-76045-7.

[8] N. Kyurkchiev, A. Iliev, S. Markov, Some Techniques for Recurrence Generating of Activation Functions: Some Modeling and Approximation Aspects, LAP LAMBERT Academic Publishing, 2017, ISBN: 978-3-330-33143-3.

[9] S. Markov, A. Iliev, A. Rahnev, N. Kyurkchiev, A note on the Log-logistic and transmuted Log-logistic models. Some applications, Dynamic Systems and Applications, **27**(3) 2018, 593–607.

[10] S. Markov, N. Kyurkchiev, A. Iliev, A. Rahnev, On the approximation of the cut functions by hyper-log-logistic function, Neural, Parallel and Scientific Computations, **26**(2) 2018, 169–182.

[11] N. Kyurkchiev, A. Iliev, Extension of Gompertz-type equation in modern science: 240 Anniversary of the birth of B. Gompertz, LAP LAMBERT Academic Publishing, 2018, ISBN: 978-613-9-90569-0.

[12] N. Kyurkchiev, A. Iliev, A. Rahnev, Some Families of Sigmoid Functions: Applications to Growth Theory, LAP LAMBERT Academic Publishing, 2019, ISBN: 978-613-9-45608-6.

[13] A. Iliev, N. Kyurkchiev, S. Markov, On the Approximation of the step function by some sigmoid functions, Mathematics and Computers in Simulation, **133** 2017, 223–234.

[14] S. Markov, N. Kyurkchiev, A. Iliev, A. Rahnev, On the approximation of the generalized cut functions of degree $p+1$ by smooth hyper-log-logistic function, Dynamic Systems and Applications, **27**(4) 2018, 715–728.

[15] N. Pavlov, A. Iliev, A. Rahnev, N. Kyurkchiev, Some software reliability models: Approximation and modeling aspects, LAP LAMBERT Academic Publishing, 2018, ISBN: 978-613-9-82805-0.

[16] N. Pavlov, A. Iliev, A. Rahnev, N. Kyurkchiev, Nontrivial Models in Debugging Theory: Part 2, LAP LAMBERT Academic Publishing, 2018, ISBN: 978-613-9-87794-2.

[17] G. W. Topp, F. C. Leone, A family of J–shaped frequency functions, Journal of American Stat. Association, **50** 1955, 209–219.

[18] M. Elgarhy, M. Nazir, F. Jamal, G. Ozel, The type II Topp–Leone generated family of diftributions: properties and applications, J. of Statistics and Management Systems, **21**(8) 2018, 1525–155.

[19] A. Hassan, M. Elgarhy, Z. Ahmad, Type II Generalized Topp–Leone Family of Distributions: Properties and Applications, Journal of data science, 2018, 20 pp.

[20] A. Mahdavi, Generalized Topp–Leone family of distributions, Journal of Biostatistics and Epidemiology, **3**(2) 2017, 65–75.

[21] H. Reyad, M. Alizadeh, F. Jamal, S. Othman, The Topp Leone odd Lindley–G family of distributions: properties and applications, Journal of Statistics & Management Systems, 2019, doi:10.1080/09720510.2018.1495157.

[22] R. Anguelov, S. Markov, Hausdorff Continuous Interval Functions and Approximations, In: SCAN 2014 Proceedings, LNCS, ed. by J.W.von Gudenberg, Springer, Berlin, 2015.

[23] R. Anguelov, S. Markov, B. Sendov, On the Normed Linear Space of Hausdorff Continuous Functions. In: Lirkov, I., et al. (Eds.): Lecture Notes in Computer Science, **3743**, Springer, 2006, 281–288.

[24] R. Anguelov, S. Markov, B. Sendov, Algebraic Operations on the Space of Hausdorff Continuous Functions. In: Bojanov, B. (Ed.): Constructive Theory of Functions, Prof. M. Drinov Academic Publ. House, Sofia, 2006, 35–44.

[25] R. Anguelov, S. Markov, B. Sendov, The Set of Hausdorff Continuous Functions - the Largest Linear Space of Interval Functions, Reliable Computing, **12** 2006, 337–363.

[26] R. Anguelov, N. Kyurkchiev, S. Markov, Some properties of the Blumberg's hyper-log-logistic curve, Biomath, **7** 2018, 8 pp.

[27] H. M. Yousof, M. Alizadeh, S. M. A. Jahanshahi, T. G. Ramires, I. Ghosh, G.G. Hamedani, The Transmuted Topp–Leone G Family of Distributions: Theory, Characterizations and Applications, Journal of Data Science, **15** 2017, 723–740.