



Mathematical Analysis of Some Reaction Networks Inducing Biological Growth/Decay Functions

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Abstract

In this work, we study some characteristics of sigmoidal growth/decay functions that are solutions of dynamical systems. In addition, the studied dynamical systems have a realization in terms of reaction networks that are closely related to the Gompertzian and logistic type growth models. Apart from the growing species, the studied reaction networks involve an additional species interpreted as an environmental resource. The reaction network formulation of the proposed models hints for the intrinsic mechanism of the modeled growth process and can be used for analyzing evolutionary measured data when testing various appropriate models, especially when studying growth processes in life sciences. The proposed reaction network realization of Gompertz growth model can be interpreted from the perspective of demographic and socio-economic sciences. The reaction network approach clearly explains the intimate links between the Gompertz model and the Verhulst logistic model. There are shown reversible reactions which complete the already known non-reversible ones. It is also demonstrated that the proposed approach can be applied in oscillating processes and social-science events. The paper is richly illustrated with numerical computations and computer simulations performed by algorithms using the computer algebra system Mathematica.

Keywords: Dynamical growth models; Logistic model; Gompertz function; Bateman equations; Sigmoidal functions; Dynamical systems; Reaction networks; Growth-decay models; Exponential growth-decay reaction chain; Brusselator; Oregonator

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1 Introduction

Chemical reaction network theory is an area of Applied Mathematics which describes the behavior of real chemical systems using mathematical models. A chemical reaction network is defined by sets of chemical species which react to form different sets of species which happens with certain speed. The aim of this work is to explore the behavior of the reactants in dependence of the rate of the reactions. The mathematical modelling is a very useful and relevant way to study the chemical reaction networks. Complex systems are typically too expensive to simulate in finer detail. Hence, there is a need for multiscale models and corresponding computational algorithms that can adaptively operate at the most coarse-grained level possible for computational efficiency. The models, which we use for modelling, rely on modelling assumptions which are the key factor for their appropriateness. These assumptions can be made from observations and as in most cases, are based on experimental data. We illustrate the method of mathematical modelling which we apply in studying the dynamical processes in B. We use "Gedankenexperiment" A to demonstrate the modelling of Free-Fall Law and the process of searching a better model which is our main goal in this work.

There are well-known reaction network realizations for a number of dynamical growth models, such as saturation, logistic, epidemic, etc.. However, there is no such realization of Gompertz-Bateman model with its reversible reaction. We discuss important links between the Gompertz and the logistic model. We also propose several reaction networks inducing dynamical models that generalize the Gompertzian one and the Verhulst Differential model. We study the reaction when the reactant is also the product of the reaction which is a modification of the classical Verhulst model not seen in the literature before.

About the contents.

We focus our attention on studying dynamical growth/decay processes in terms of reaction networks. The current work is structured as follows:

In **Chapter 2**, we consider basic reaction kinetics models, including saturated and logistic growth/decay models. Besides the classical study of growth functions, the relationship between growing and decaying species in a reaction is obtained as an idea that the product of the reaction increases at the expense of the decaying reactant. We study the Verhulst Logistic Differential model, extend it and generalized it.

In **Chapter 3**, we focus our attention on Gompertz Growth model which is studied in details and extended to Gompertz-Bateman Growth/Decay Model, its solutions and generalizations are shown.

In **Chapter 4**, we present some applications of reaction networks in studying oscillating chemical reactions of the brusselator and the Oregonator.

One of the basic catalytic and autocatalytic reactions are extended and some natural generalizations and modifications induced by the proposed networks, are formulated and discussed. There is another view of the Verhulst logistic growth model which differs from the classical study in [11] with modification of the species which is a catalyst as species with a constant concentration. We apply the idea of changing the reaction's equations when assuming that some of the reactants are with a constant concentration through the reaction.

2 Reaction Kinetics

2.1 Modelling Saturated Growth/Decay

A main tool used in the analysis of chemical reaction networks is the Law of Mass Action.

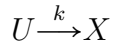
Law of Mass Action:

The rate of a reaction is proportional to the product of the concentrations of the reactants.

Using this law, it is obvious how to write the differential equations,

which show the behavior of the concentrations of the reactants and the products of the reactions over time.

The first and the most common chemical reaction, which we can look at is the reaction:



Here we have a transformation from the substance U to the substance X with a constant rate k . From the Law of Mass Action, we can write down the following system of differential equations:

$$\begin{cases} x' = ku \\ u' = -ku \end{cases} \quad (1)$$

The solution x, u of system (1) can be easily found as:

$$\begin{aligned} u(t) &= u(0)e^{-kt} \\ x(t) &= x(0) + u(0)(1 - e^{-kt}) \end{aligned}$$

The graphics shows saturation curve for x .

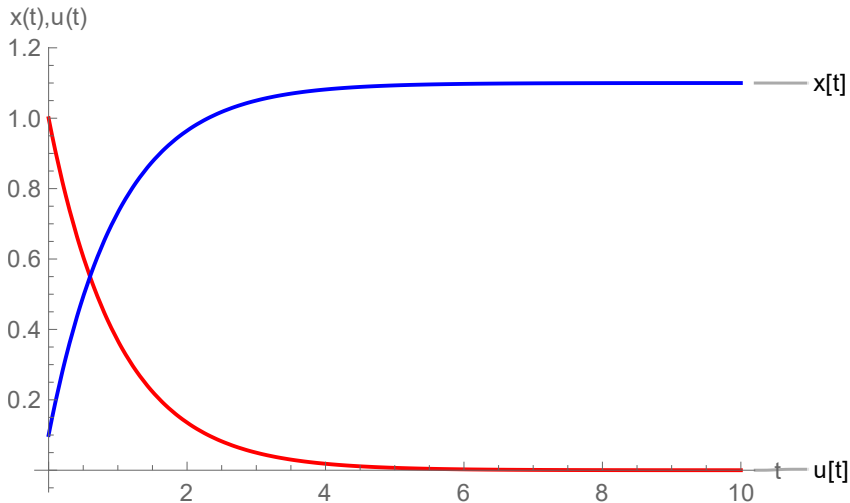
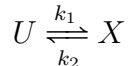


Figure 1: The solution x, u of system (1)

Reversible Reaction. Now, let us try to solve the case when the reverse reaction is possible.



It could be treated analogously. We have the system:

$$\begin{cases} u' = k_2 x - k_1 u \\ x' = -k_2 x + k_1 u \end{cases} \quad (2)$$

The solution for u, x of the system (3) for $u(0) = u_0$ and $x(0) = x_0$ is:

$$\begin{aligned} u(t) &= \frac{e^{-(k_1+k_2)t}(k_2 u_0 - k_1 x_0 + e^{(k_1+k_2)t} k_1 (u_0 + x_0))}{k_1 + k_2} \\ x(t) &= \frac{e^{-(k_1+k_2)t} k_1 x_0 + k_2 (u_0 - e^{-(k_1+k_2)t} u_0 + x_0)}{k_1 + k_2} \end{aligned} \quad (3)$$

and the behavior of the reactants will depend on the rates of the reactions k_1 and k_2 . When the rate of the right reaction is bigger, the solution becomes the one plotted on Figure 2. The opposite case is shown on Figure 3:

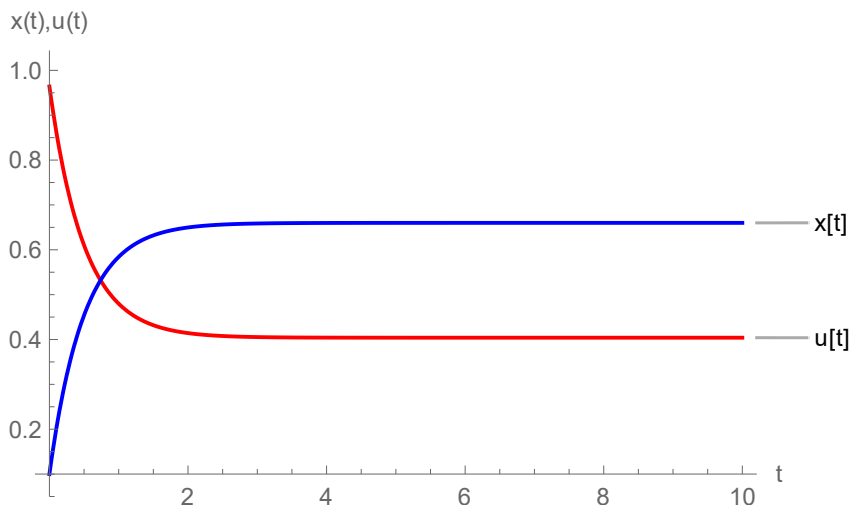


Figure 2: The solution for x, u of system (3) when $k_1 > k_2, k_1 = 1.2, k_2 = 0.8$

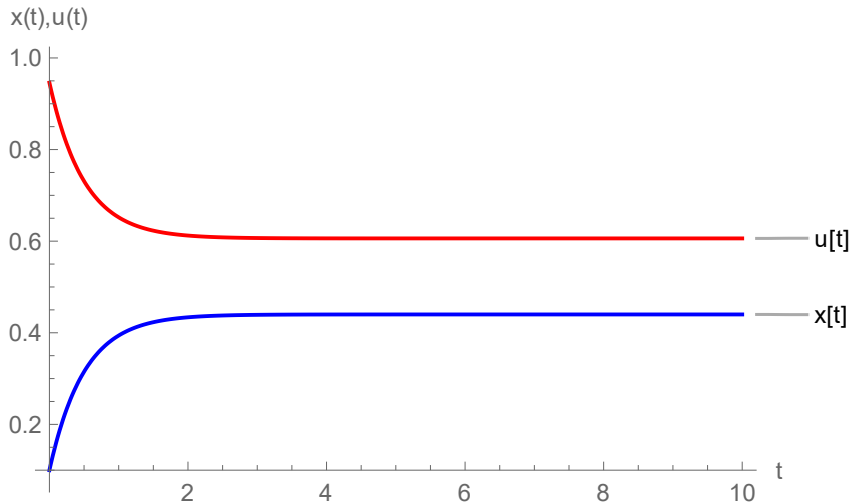


Figure 3: The solution for x, u of system (3) when $k_2 > k_1$, $k_1 = 0.8$, $k_2 = 1.2$

The relation between the rate constants, in this case, means that while X grows, U decays. Note that the saturation growth process has no lag phase. Growth processes with a lag phase are typical for bio-chemical reactions involving functional proteins.

2.2 Modelling Logistic Growth/Decay

2.2.1 Simple Catalytic Reaction

The behavior with only one substrate which turns into another one is simple and easy to predict. The more interesting reactions are those that have more participating substances. Here the species X has the role of a catalyst (a substance that causes or accelerates a chemical reaction without being affected). It is a reactant and a product at the same time. If it doesn't occur in another reaction, it remains constant throughout the reaction.

Consider the following autocatalytic reaction network:



we have the dynamical system:

$$\begin{cases} x' = 0 \\ u' = -kux \end{cases} \quad (5)$$

The solution of the system (5) is:

$$\begin{aligned} x(t) &= C_1 = \text{const} \\ u(t) &= e^{-kC_1 t} C_2 \end{aligned}$$

where C_1 and C_2 are constants which depend on the initial condition of the substances. Let the values at the time $t = 0$ are $x(0) = x_0$ and $u(0) = u_0$. We have that $C_1 = x_0$ and $C_2 = u_0$:

$$\begin{aligned} x(t) &= x_0 \\ u(t) &= e^{-kx_0 t} u_0 \end{aligned}$$

We can see graphically the behavior of the solution at Figure 4. We are assuming that we have the initial conditions $x_0 = 0.1$ and $u_0 = 1$:

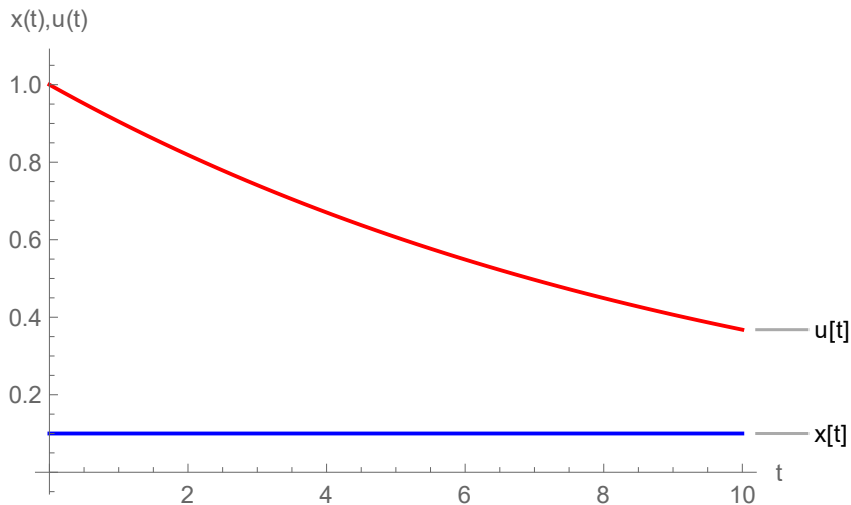
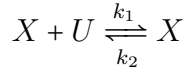


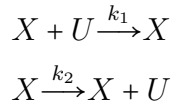
Figure 4: The graphics of the solution x, u of the system (4).

2.2.2 Reversible Catalytic Reaction

Now let us turn our attention to the situation when the reverse reaction is possible. It could be treated in the same way as the non-reversible reaction:



We will decompose the reaction into two different reactions:



We have the dynamical system:

$$\begin{cases} x' = 0 \\ u' = -k_1 u x + k_2 x \end{cases} \quad (6)$$

The solution of this system (6) is:

$$\begin{aligned} x(t) &= \text{const} = C_1 \\ u(t) &= \frac{k_2}{k_1} + e^{-k_1 t C_1} C_2 \end{aligned}$$

with $x(0) = x_0$ and $u(0) = u_0$:

$$\begin{aligned} x(t) &= x_0 \\ u(t) &= \frac{k_2}{k_1} + \left(u_0 - \frac{k_2}{k_1} \right) e^{-x_0 k_1 t} \end{aligned}$$

Here the solutions depend on the values of k_1 , k_2 and u_0 .

If $k_1 u_0 > k_2$, the function $u(t)$ will decrease.

If $k_1 u_0 < k_2$, the function $u(t)$ will increase.

Let us take again $x_0 = 0.1$ for the initial conditions. If $k_1 u_0 > k_2$ we have the same situation as in Figure 4:

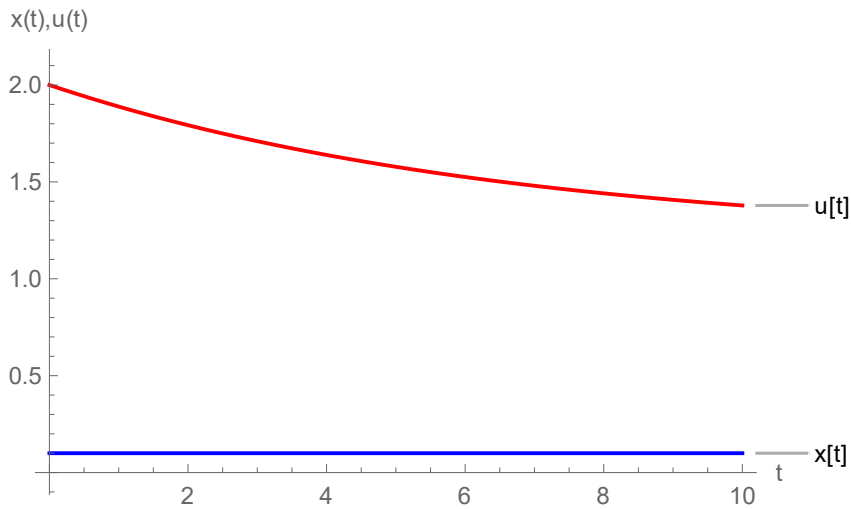


Figure 5: The graphics the of reaction when $k_1 u_0 > k_2$ and $k_1 = 1.5, k_2 = 1.8, u_0 = 2$.

When the rate of the reverse reaction is bigger, we have the opposite:

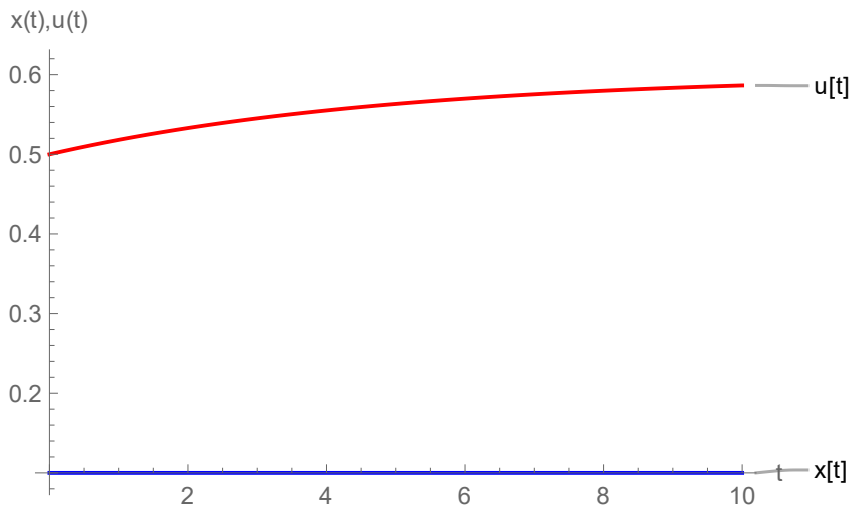


Figure 6: The graphics of the reaction when $k_2 > k_1 u_0$ and $k_1 = 2, k_2 = 1.2, u_0 = 0.5$

In the case $k_2 > k_1 u_0$ the solution grows and in this simple reaction we observe bifurcation A.

2.2.3 Verhulst Logistic Differential Model

Non-Reversible Reaction. Let us look at the reactions that have more products and the catalyst X replicates itself throughout the reaction. We will focus our research on especially these special kind of reactions - autocatalytic reactions. The autocatalytic reaction is a process whereby a chemical is involved in its own production. Let us look at the simple autocatalytic reaction:



where X and U are some substances and k is the rate constant of the reaction. The reaction (7) induces a logistic growth model. [4] One more possible interpretation could be the following: U transits(converts) into X under the influence of the catalyst X itself. Using the Mass Action Law, reaction network is written in terms of rate equations for the concentrations u, x as follows [15]:

$$\begin{cases} x' = kux \\ u' = -kux \end{cases} \quad (8)$$

The solution of the system is:

$$x(t) = C_1 + \frac{C_1}{-1 + e^{ktC_1 + C_1C_2}}$$

$$u(t) = -\frac{C_1}{-1 + e^{ktC_1 + C_1C_2}},$$

where C_1 and C_2 are constants which depend on the initial condition of the substances. We can see graphically the behavior of reactants on Figure 7. We are assuming that we have the initial conditions $x(0) = x_0 = 0.1$ and $u(0) = u_0 = 1$.

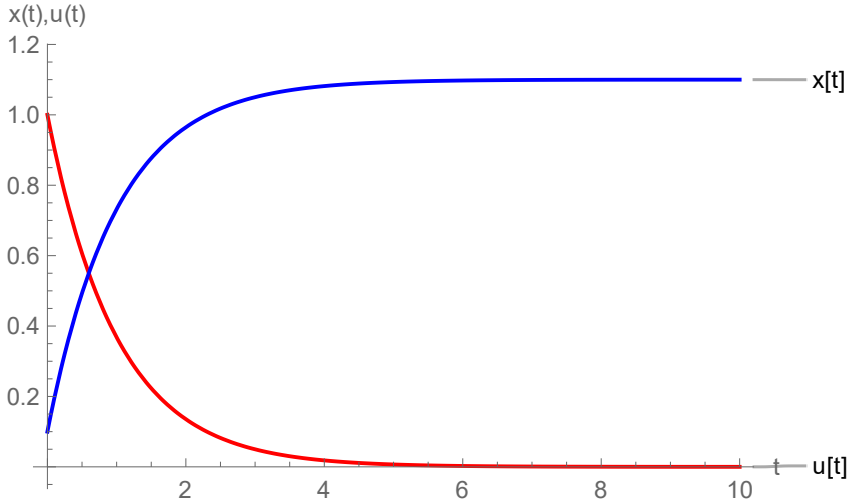


Figure 7: The graphics of the reaction when $x(0) = x_0 = 0.1$, $u_0 = u_0 = 1$ and $k = 1$.

From the rate equations we can easily notice that:

$$u' + x' = 0$$

Let $u(0) = u_0$ and $x(0) = x_0$. Therefore $u + x = u_0 + x_0 = \text{const} = a$. Substituting $u = a - x$ in the equation for x , we have: $x' = kxu = kx(a - x)$, which is the familiar Verhulst differential equation. It is usually written with normalized constant $k := \frac{k}{a}$ as follows:

$$\frac{dx}{dt} = \frac{k}{a}x(a - x) = kx \left(1 - \frac{x}{a}\right) \quad (9)$$

This is the well-known Verhulst logistic differential model. [17]

The solution of (9) passing through the point $(0, x(0) = \frac{a}{2})$ is the logistic sigmoidal function:

$$x_0(t) = \frac{a}{1 + be^{-kt}}, b = \frac{a - x_0}{x_0} = 1$$

with a “jump” at the origin and asymptotes: $\lim_{t \rightarrow -\infty} x_0(t) = 0$ and $\lim_{t \rightarrow \infty} x_0(t) = a$, where usually $a = 1$. In the general case where the

“jump” is at given point γ , the logistic Verhulst function is written as:

$$x_\gamma(k; t) = \frac{1}{1 + e^{-k(t-\gamma)}} \quad (10)$$

The logistic function has an inflection at $(\gamma, \frac{1}{2})$, its graph is centrally symmetric and its slope at γ is $\kappa = \frac{k}{4}$. The so-called lag time in the Verhulst logistic model is $t_{\text{lag}} = \gamma - \frac{2}{k}$. The graph of the logistic function is a sigmoid and looks like presented on Figure 8:

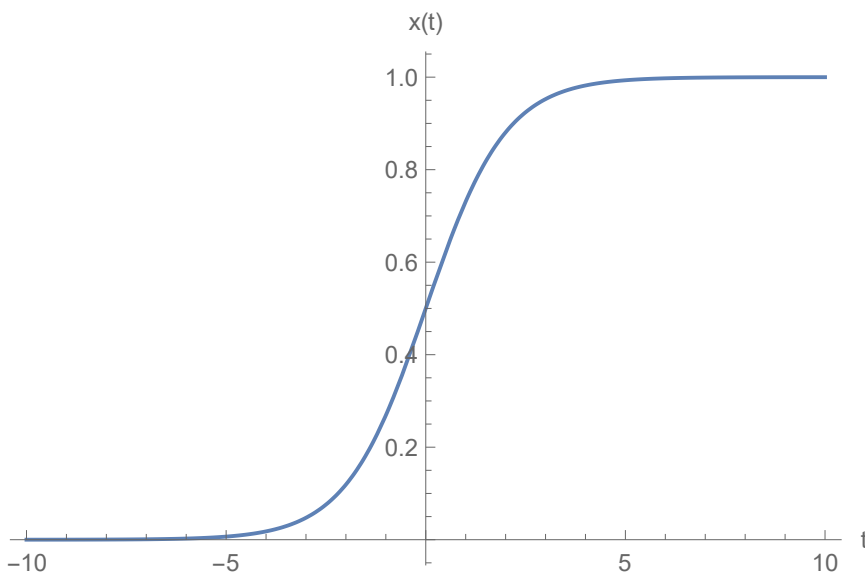


Figure 8: Verhulst logistic function with $\gamma = 0$ and $k = 2$.

To extend the model, we will continue with studying the process when the reversible reaction also occurs.

Reversible Reaction. When the reaction is also possible in the reverse direction, we have:



The case when the substance U is maintained at a constant concentration u is discussed in [11]. We will modify the reaction network (11) with adding the species U at the right side of the reaction. In such way, there is no need of an assumption for a constant concentration of the species U, it comes naturally from the reaction:



We will call and study further the operator induced by this reaction as "Gompertz differential operator". [9] [10] The Law of Mass Action applied to this reaction gives the rate of reaction as

$$x' = k_1 u x - k_2 x^2 u \quad (13)$$

From here, we have that the final nonzero steady state when $t \rightarrow \infty$ will be:

$$x_\infty = \frac{k_1}{k_2}$$

The zero steady state is unstable by inspection.

Next, we will focus on a more common situation - when the concentration of the reactant U is not constant in (11).

Using the Law of Mass Action,

$$\begin{cases} x' = k_1 \cdot u \cdot x - k_2 \cdot x^2 \\ u' = -k_1 \cdot u \cdot x + k_2 x^2 \end{cases} \quad (14)$$

The behavior of substances depends on the rate constants. Let us look at some cases for the constants k_1 and k_2 .

We have already looked at the case when the reaction is not reversible ($k_2 = 0$), so we consider the following cases.

Case 1: When only the reverse reaction happens, $k_1 = 0$ and the reaction system looks different.



We could look at it as the right reaction with the following equations:

$$\begin{cases} x' = -k_2 \cdot x^2 \\ u' = k_2 \cdot x^2 \end{cases}$$

The solution is:

$$\begin{aligned} x(t) &= \frac{x_0}{k_2 t x_0 + 1} \\ u(t) &= u_0 + x_0 - \frac{x_0}{1 + k_2 t x_0} \end{aligned} \quad (16)$$

In this case, the equations for x and u are exchanged and the graph will look like:

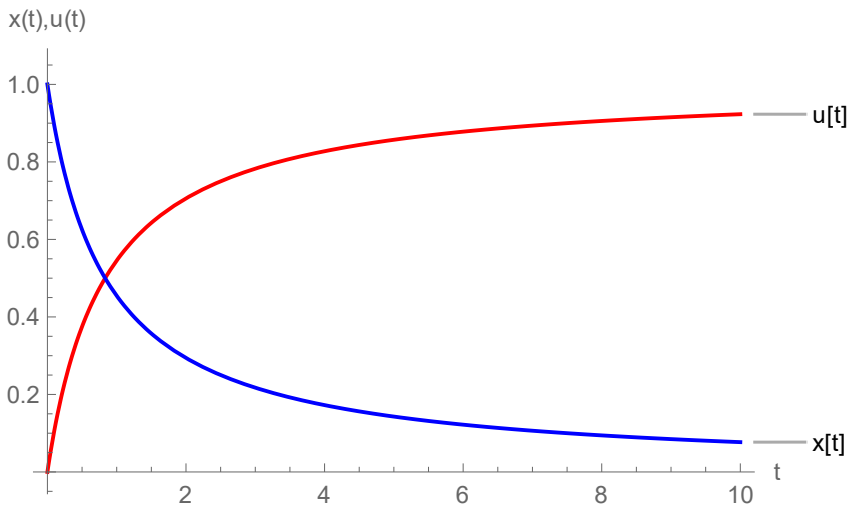


Figure 9: The graphics of the reaction (15) when $k_1 = 0, k_2 = 1.2$.

Case 2: In the case when $k_1 > k_2$ we have the same graphics as the previous models, shown on Figure 7.

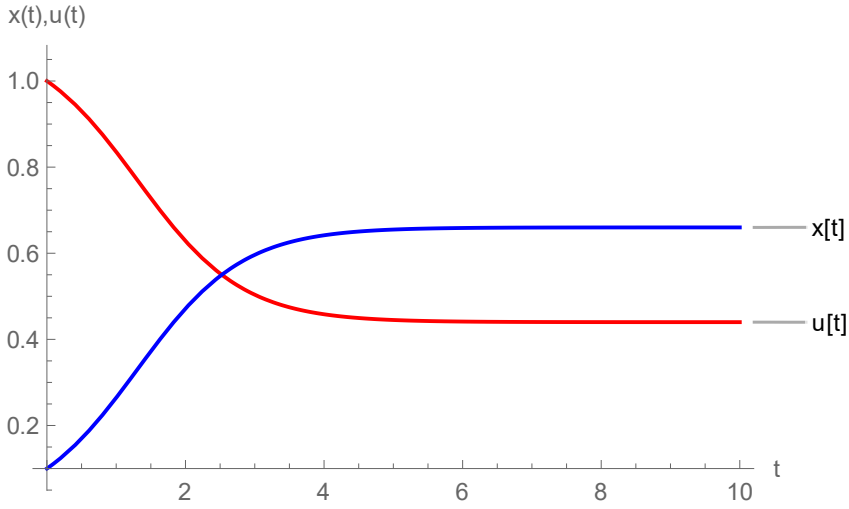


Figure 10: Time evolution of the reactants in the case $k_1 > k_2$ where $k_1 = 1.2, k_2 = 0.8$

Case 3: Another case, which is needed to be shown, is the case, when the reversible reaction happens with higher rate, i.e. $k_1 < k_2$.

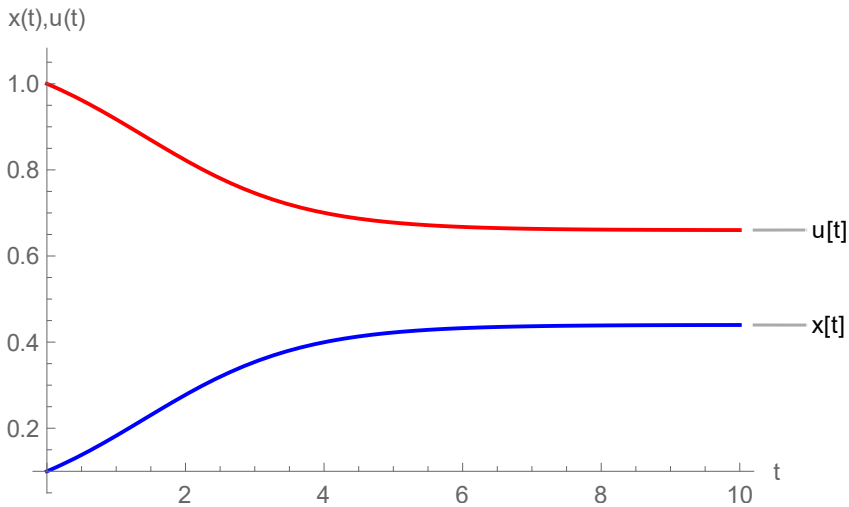


Figure 11: Time evolution of the reactants in the case $k_1 < k_2$ where $k_1 = 0.8, k_2 = 1.2$

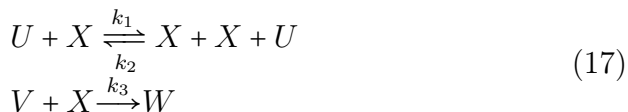
Finally, we can solve the system implied by the reaction network $U + X \xrightleftharpoons[k_2]{k_1} X + X$ exactly for the concentrations of X and U. We have:

$$x(t) = \frac{e^{k_1 C_1 (t+C_2)} k_1 C_1}{-1 + e^{k_1 C_1 (t+C_2)} (k_1 + k_2)};$$

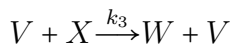
$$u(t) = \frac{-1 + e^{k_1 C_1 (t+C_2)} C_1}{-1 + e^{k_1 C_1 (t+C_2)} (k_1 + k_2)},$$

where C_1 and C_2 are constants.

Suppose that besides the reaction (17) we have one more reaction in the system where X is used up in the production of W([11]):



Again, if we assume that V is with a constant concentration this leads us to the reaction where V is a catalyst and:



The solutions and the conclusions we can make for X don't differ from these made in [11]. We have:

$$x' = k_1 u x - k_2 x^2 u - k_3 v x = (k_1 u - k_3 v) x - k_2 x^2 u$$

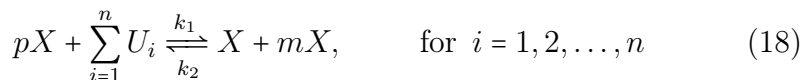
Here $k_1 u$ is the unit production rate of x and $k_3 v$ is the unit loss rate. If $k_1 u > k_3 v$ the steady state $x = 0$ is unstable and as $t \rightarrow +\infty$, $x = \frac{k_1 u - k_3 v}{k_2}$ which is stable. On the other hand, if $k_1 u < k_3 v$ $x = 0$ is stable which is not surprising since the inequality implies that the loss rate is greater than the production rate.

The changes we have made with adding the reactant as a product of the reactions didn't affect the essential of the reactions studied in [11], only gave more natural interpretation for the reactions.

There are a lot of examples of chemical reactions, most commonly in Biology, which have a reaction network of the form $X + U \longrightarrow X + X$. An interpretation of this reaction could be that a substance makes a reaction with another substance and as a result of the interaction, the first substance is either activated, doubled or just increased. Here, the substance U can be interpreted as an activator or catalyst for the other substance. We can see such reactions in the action of the enzymes. An example is the enzyme trypsin - when the pancreas is stimulated by cholecystokinin, the enzyme is then secreted into the first part of the small intestine via the pancreatic duct. Once in the small intestine, the enzyme enteropeptidase activates trypsinogen into trypsin by proteolytic cleavage. Another popular example is that of the prions (infectious agents composed entirely of protein material) which attack a protein, the prion converts the protein into itself. Only in this case, there is no other substrate, which is released, because the prions change the configuration and fold of the attacked molecule without removing anything from it. They change only the spatial structure of the protein.

In general, this reaction has an important part of the Biology science and occurs in every system with proteolytic activation of enzymes.

Generalized Verhulst Logistic Differential Model. We can make a generalization of the Verhulst Logistic Differential Model. Let us look at the reaction in which we have n substances U which transits and duplicate p substances X $m + 1$ times, where $p \geq 0$ and $p < m$. The reactions schemes will look like:



We obtain the following system:

$$\begin{cases} x' = k_1 x^p \prod_{j=1}^n u_j - k_2 x^{m+1}, \\ u'_i = -k_1 x^p \prod_{j=1}^n u_j + k_2 x^{m+1}, \end{cases} \quad i = 1, 2, \dots, n$$

From the system it is easy to see that for every $i = 1, 2, \dots, n$

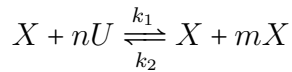
$$x' + u'_i = 0,$$

implying $u_i + x = c_i$, where $c_i > 0$ are constants. We can obtain the following equation for this general case:

$$x' = k_1 x^p \prod_{i=1}^n (c_i - x) - k_2 x^{m+1}$$

We can easily obtain the non-reversible reaction by substituting $k_2 = 0$ in 18 .

For $p = n = m = 1$ we have the already viewed Verhulst model. For $n = m, p = 1$ and $U_i = U$ for $i = 1, 2, \dots, n$ we have:



Let us look at the model when $p = 2, n = 1, m = 2, U_1 = U$ and $k_2 = 0$.



We have the following system:

$$\begin{cases} x' = k_1 x^2 (c - x), \\ u' = -k_1 x^2 (c - x) \end{cases} \tag{20}$$

Using the initial conditions $x(0) = x_0$ and $u(0) = u_0$ we find the constant $c = x_0 + u_0$. For the graphic of the solution of the system 20:

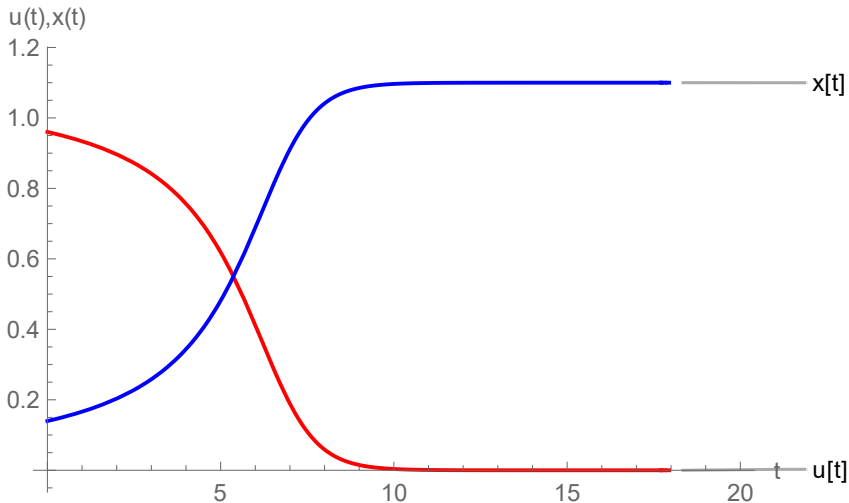


Figure 12: Time evolution of the reactants in the case $k_1 < k_2$ where $k_1 = 0.8, k_2 = 1.2$

We focused on exactly this reaction because it participates in brusselator model which we will define in (4.1) and study thereafter.

A possible “biochemical” interpretation of model (18) is as follows: the model is a one that describes the interaction between several species, such as various types of foods and other environmental resources (water, air, light, etc). In chemistry, it is unlikely that more than three species interact simultaneously. However, in models related to biology and social sciences, this restriction can be relaxed.

3 The Gompertz Growth Model:

3.1 Overview

We will focus our attention on the famous Gompertz growth model. The Gompertz model is a type of mathematical model for a time series and is named after Benjamin Gompertz (1779-1865). It is a sigmoid function which describes growth as being slowest at the start and end of a given time period.

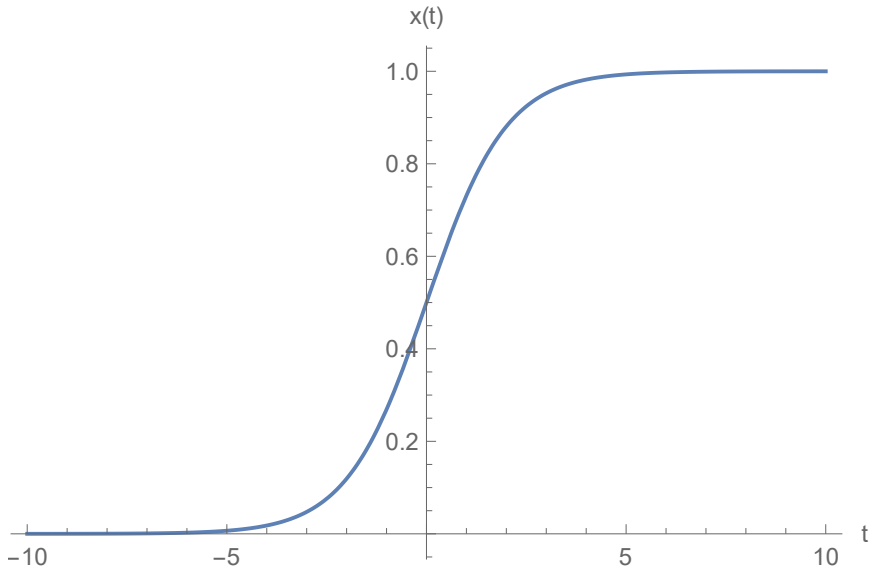


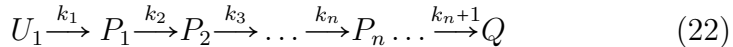
Figure 13: The sigmoid function serves as the basis of the Gompertz function, in which initial growth is rapid followed by a levelling-off

It is the same sigmoid curve we saw when dealing with the Verhulst Growth model. The function was originally designed to describe human mortality but after that has been modified to be applied in biology, with regards to detailing populations. The Gompertz growth function is the solution of the dynamical equation for the classical Gompertz model (where $x = x(t)$):

$$x' = kx(c - \ln x) \tag{21}$$

The Gompertz model can be formulated in terms of a reaction network. Let us look at a reaction network which consists of three species and two reaction equations. Let us again have a "catalyst/resource" U that declines and tends to 0 with $t \rightarrow +\infty$, and a sigmoidally growing species X catalyzed by species U. The two reactions determine the independent behavior of the catalyst U from the catalyzed species X. We could look at the species U as a first unit of the Bateman exponential

decay chain:



We will focus our attention on a general class of growth-decay models where the resource species is replaced by some consecutive species appearing further in the Bateman chain. An important advantage of the proposed class of models is the prolonged lag phase of the sigmoidally growing species X - a property often needed when fitting biological data of growth-decay processes, studied in areas like microbiology, marine ecology, tumor research, etc. [1], [5], [6], [14].

The classical Gompertz model with catalyst U and catalyzed species X looks like:



For the rate of concentrations, we have the following system:

$$\begin{cases} p' = k_1 u \\ u' = -k_1 u \\ x' = k u x \end{cases} \quad (24)$$

Let us assume the following initial conditions:

$$u(0) = u_0 > 0, p(0) = p_0 \geq 0, x(0) = x_0 > 0 \quad (25)$$

Proposition 1: From (24) and (25) we can conclude the following properties for the variables $x, u \in [0, +\infty]$:

•

$$\frac{k}{k_1} u = c - \ln x, c = \frac{k}{k_1} u_0 + \ln x_0 \quad (26)$$

- the variable x satisfies the classical Gompertz equation

Proof. Let us ignore the equation for p from (24). Then we have

$$u' = k_1 u, x' = k u x$$

Therefore,

$$\begin{aligned} \frac{u'}{k_1} + \frac{x'}{kx} &= 0, \\ \implies \left(\frac{1}{k_1} u + \frac{1}{k} \ln x \right) &= \text{const} = b, \\ \implies \frac{1}{k_1} u &= b - \frac{1}{k} \ln x \end{aligned}$$

From here, when we multiply both sides by k and use the initial conditions, we get (26).

Substituting u from (26) in the equation $x' = k u x$, we obtain the familiar Gompertz differential equation (21)

$$x' = k_1 x (c - \ln x), x(0) = x_0 \quad (27)$$

for the concentration x of the growth species X. We assume that $\lim_{t \rightarrow \infty} x = 1$ determines the constant c as $c = 0$. The solution of the equation (27) is the familiar Gompertz function

$$x(t) = x_0^{e^{-k_1 t}} \quad (28)$$

The graph of the solution $x = x(t)$ is shown on Figure 14

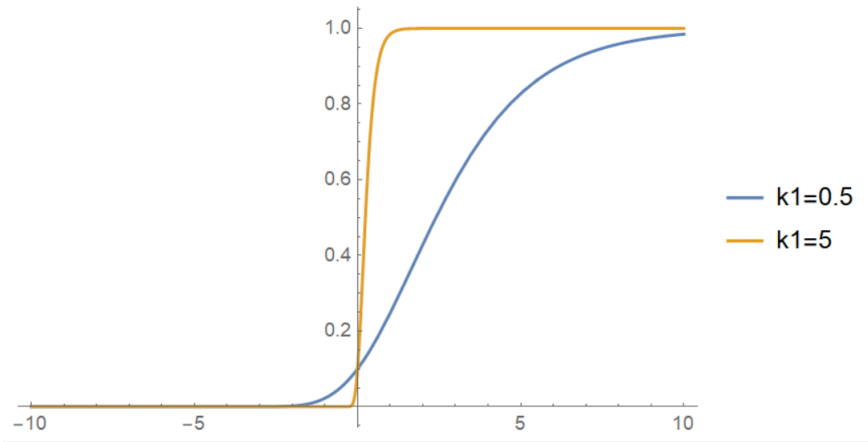
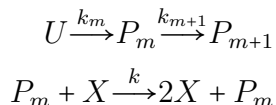


Figure 14: The graphic of the Gompertz function for $k_1 = 0.5$ and $k_1 = 5$, $x_0 = 0.1$

3.2 A Gompertz-Bateman Growth/Decay Model

Let us look at the classical Gompertz model including reactions between species U, X and P (23). It is easy to see that the evolution of species U does not depend on species X. It is shown mathematically in (24). Comparing this model with Verhulst Logistic Model, we can see that there the species U vanishes as a catalyst, so species X uses U as a resource and the two species are strongly connected. Differently, in the Gompertz model X does not influence the (bio)mass of U during the process of reproduction, that is X does not consume U as food or resource but uses U merely as a catalyst. The reaction $U \xrightarrow{k_1} P$ says that at time moment t_0 a concentration quantity u_0 instantly appears in the environment of the growing species X and the process immediately starts. Such an instant appearance of U, thereby homogeneously distributed in the volume, seems not very realistic. The more realistic view of the process will be that the catalyst species arrives smoothly in time in the form of a wave, possibly starting from an arbitrarily small value greater or equal to zero. This scenario is possible if the

initial species U changes to another species or a set of other species smoothly after a couple of reactions and then the new species participate in the reaction with X as a catalyst. The Bateman chain (22) suggests that, instead of the first species U in the chain, we may use some species next in the chain to play the role of a catalyst for the growing species X . We will call the m -th component of the Bateman chain for the catalytic reaction with X the Gompertz-Bateman model of order m .



We will focus our attention on the Gompertz-Bateman model of order 1 [10]:



Let for briefness denote $P_1 \equiv P$. As usually, assume that species U , P and X are homogeneously distributed in a fixed volume and denote their concentrations (masses) respectively by u, p and x . The reaction (29) implies the following dynamical system:

$$\begin{cases} u' = -k_1 u \\ p' = k_1 u - k_2 p \\ x' = k x p \end{cases} \tag{30}$$

Let us assume the following initial conditions:

$$u(0) = u_0 > 0; p(0) = p_0 \geq 0, x(0) = x_0 > 0 \tag{31}$$

Proposition 2: [10] From (30) and (31) we can conclude the following properties for the variables $x, u \in [0, +\infty]$:

$$\frac{k}{k_2}(p + u) = c - \ln x, c = \frac{k}{k_2}(u_0 + p_0) + \ln x_0 \tag{32}$$

Proof. From the equation in (30) we notice that:

$$\begin{aligned}
 u' + p' &= -k_2 p = -k_2 \frac{x'}{kx} \\
 \implies (u + p)' &= -k_2 \left(\frac{1}{k} \ln x \right)' \\
 \implies u + p + k_2 \frac{1}{k} \ln x &= \text{const} = C_1 \\
 \implies \frac{k}{k_2} (p + u) &= c - \ln x
 \end{aligned}$$

Using the initial conditions from (31) we obtained the desired relationship.

Reversible Reaction. In addition to the non-reversible case (29) considered in [10], let us assume that the reversible reaction is also possible with some rate constant l and we have the following system:



The reactions (33) implies the following dynamical system:

$$\begin{cases}
 u' = -k_1 u \\
 p' = k_1 u - k_2 p \\
 x' = kxp - lx^2 p
 \end{cases} \tag{34}$$

Let us have the same initial conditions.

$$u(0) = u_0 > 0; p(0) = p_0 \geq 0, x(0) = x_0 > 0 \tag{35}$$

Then, we can conclude the following relations between the variables $x, u \in [0, +\infty]$:

Proposition 2*:

$$\begin{aligned}\frac{k}{k_2}(u + p) &= c - \ln(x) + \ln(k - lx), \\ c &= \frac{k}{k_2}(u_0 + p_0) + \ln(x_0) - \ln(k - lx_0)\end{aligned}\tag{36}$$

Proof. From the equation in (34) we notice that:

$$\begin{aligned}u' + p' &= -k_2p = -k_2 \frac{x'}{kx - lx^2} \\ \implies (u + p)' &= -k_2x' \left(\frac{1}{kx} + \frac{l}{k} \frac{1}{k - lx} \right) \\ \implies (u + p)' &= \left(-\frac{k_2}{k} \ln x + \frac{k_2}{k} \ln(k - lx) \right)' \\ \implies u + p &= c_1 - \frac{k_2}{k} \ln x + \frac{k_2}{k} \ln(k - lx) \\ \implies \frac{k}{k_2}(u + p) &= c - \ln x + \ln(k - lx)\end{aligned}$$

,where $c_1 = \text{const}$. Using the initial conditions from (35) we obtained the desired relationship.

3.3 Solutions to Gompertz-Bateman Growth/Decay Model of Order 1

We will find the exact solutions for the variables u, p and x using the dynamical system (30) and the initial conditions (31). The solutions for functions u and p are well-known as being part of the Bateman equations [2]. For u from the equation $u' = -k_1u$ we have:

$$u(t) = u_0 e^{-k_1 t}\tag{37}$$

For p from the equation $p' = k_1u - k_2p = k_1u_0 e^{-k_1 t} - k_2p$ using Laplace Transform(see Appendix D) we get:

$$p(t) = \begin{cases} \frac{k_1 u_0}{k_2 - k_1} (e^{-k_1 t} - e^{-k_2 t}) + p_0 e^{-k_2 t}, & k_1 \neq k_2 \\ e^{-k_1 t} (p_0 + k_1 u_0 t), & k_1 = k_2 \end{cases}\tag{38}$$

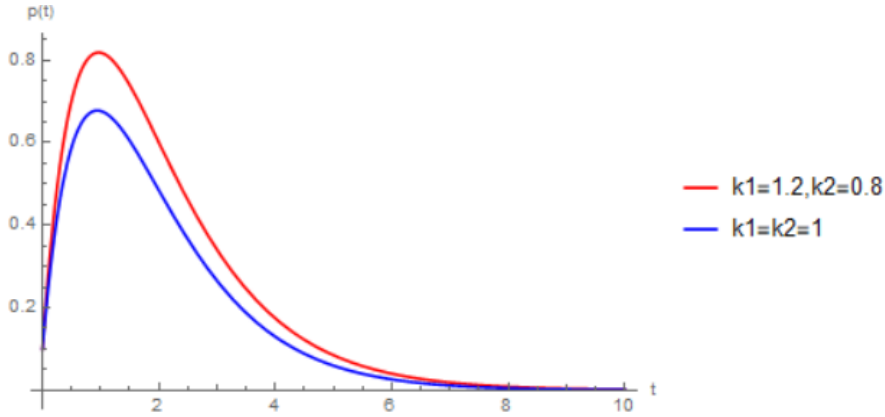


Figure 15: The function p for $k_1 = 1.2, k_2 = 0.8, p_0 = 0.1, u_0 = 1.74$ (the red one) and $k_1 = k_2 = 1, p_0 = 0.1, u_0 = 1.74$ (the blue one)

After we have the exact solution for p from (38) from the equation $x' = kxp$ and the Picard-Lindelöf theorem we can conclude that there is an unique solution for the variable x in the interval $[0; +\infty)$. Then, the Gompertz-Bateman model can be formulated as a differential equation for the growth function x :

$$x' = kxp = \begin{cases} kx\left[\left(\frac{k_1}{(k_2-k_1)}\right)u_0(e^{-k_1t} - e^{-k_2t}) + p_0e^{-k_2t}\right], & k_1 \neq k_2 \\ kxe^{-k_1t}(p_0 + k_1u_0t), & k_1 = k_2 \end{cases} \quad (39)$$

Expression (39) induces the following growth rate per capita (also known as Gompertz “mortality law”):

$$\frac{x'}{x} = kp = \begin{cases} k\left[\left(\frac{k_1}{(k_2-k_1)}\right)u_0(e^{-k_1t} - e^{-k_2t}) + p_0e^{-k_2t}\right], & k_1 \neq k_2 \\ ke^{-k_1t}(p_0 + k_1u_0t), & k_1 = k_2 \end{cases} \quad (40)$$

For the exact solution of the equation, we can substitute the solutions

for u and p and using $x' = kxp$ we easily obtain the solution for x [10]:

$$x(t) = \begin{cases} x_0 e^{\frac{ke^{-(k_1+k_2)t}(-e^{k_2 t}k_2 u_0 + e^{(k_1+k_2)t}(k_1-k_2)(p_0+u_0) + e^{k_1 t}(k_2 p_0 + k_1(-p_0+u_0)))}{k_2(k_1-k_2)}}, & k_1 \neq k_2, \\ x_0 e^{\frac{k(p_0+u_0 - e^{-k_1 t}(p_0+u_0+k_1 u_0 t))}{k_1}}, & k_1 = k_2. \end{cases} \quad (41)$$

We will show the graphic of x for different initial conditions for x_0 , $x_0 = 0.1$ and $x_0 = 0.01$. For the other parameters and initial conditions we will use the relation (32) $c = \frac{k}{k_2}(u_0 + p_0) + \ln x_0$:

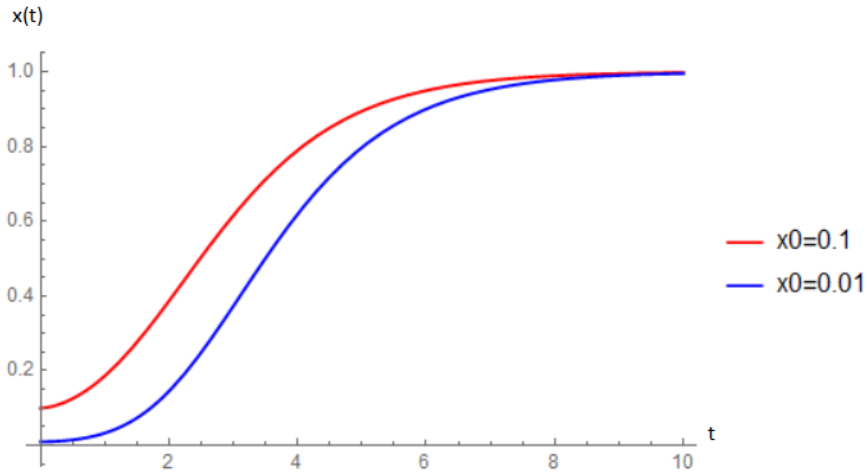


Figure 16: The solution x for $k_1 = 1.2, k_2 = 0.8, p_0 = 0.1, u_0 = 1.74, x_0 = 0.1$ (the red one) and for $k_1 = 1.2, k_2 = 0.8, p_0 = 0.1, u_0 = 1.74$ and $x_0 = 0.01$ (the blue one)

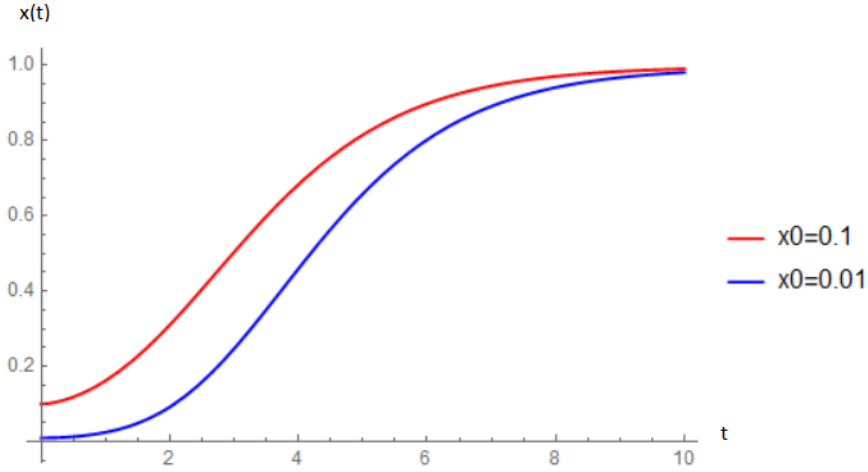


Figure 17: The solution x for $k_1 = k_2 = 0.8, p_0 = 0.1, u_0 = 1.74, x_0 = 0.1$ (the red one) and for $k_1 = k_2 = 0.8, p_0 = 0.1, u_0 = 3.58$ and $x_0 = 0.01$ (the blue one)

Solution to the Reversible Reaction. In addition to the non-reversible reaction considered in [10], we discuss next the reversible reaction. For the reversible reaction, we have the same solution for p . We use the equation for x $x' = kxp - lx^2p$ to find the exact solution.

$$x(t) = \begin{cases} \frac{k}{1 + e^{\frac{-k(p_0+u_0)}{k_2} + \frac{k(e^{-k_1 t} u_0 + e^{-k_2 t(-k_2 p_0 + k_1(p_0+u_0))})}{k_2}}}, & k_1 \neq k_2, \\ \frac{e^{\frac{k(u_0+p_0)}{k_1}} k x_0}{e^{\frac{k(u_0+p_0)}{k_1}} l x_0 + (k-l x_0) e^{\frac{-k_1 t k(p_0+u_0+k_1 t u_0)}{k_1}}}, & k_1 = k_2. \end{cases} \quad (42)$$

We found the appropriate values for the initial conditions.

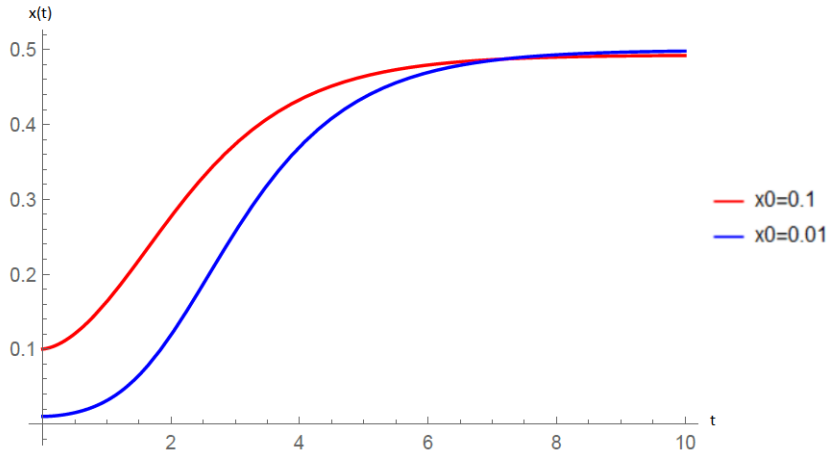


Figure 18: The solution x for $k_1 = 1.2, k_2 = 0.8, p_0 = 0.1, u_0 = 1.51, x_0 = 0.1$ (the red one) and for $k_1 = 1.2, p_0 = 0.1, u_0 = 3.43$ and $x_0 = 0.01$ (the blue one)

The Figure 19 shows the solution when $k_1 = k_2$.

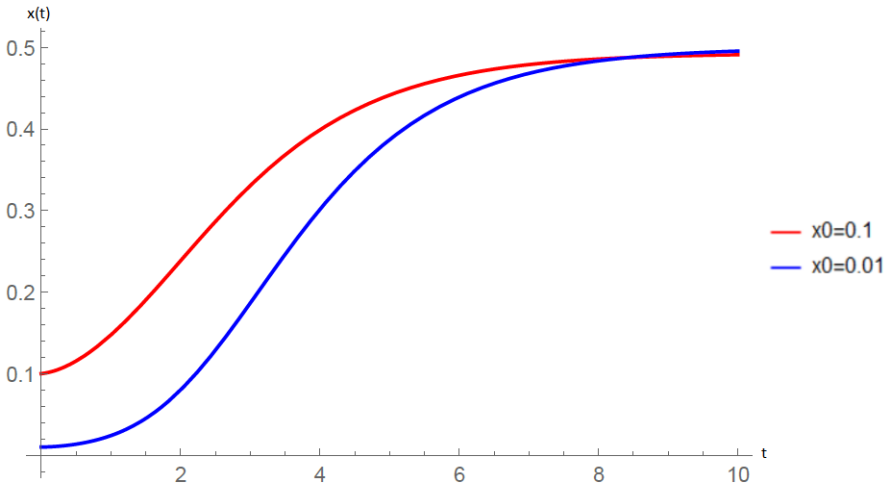
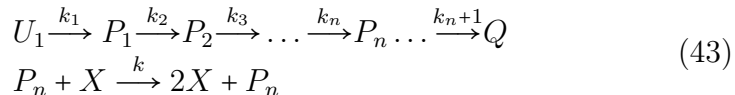


Figure 19: The solution x for $k_1 = k_2 = 0.8, p_0 = 0.1, u_0 = 1.51, x_0 = 0.1$ (the red one) and for $k_1 = k_2 = 0.8, p_0 = 0.1, u_0 = 3.43$ and $x_0 = 0.01$ (the blue one)

Generalized Gompertz-Bateman model of order n . With the initiation of the Gompertz-Bateman model of order 1 we separate the reactions for the declining species(resources) from the growing species X . Using this idea, we can substitute the catalyst in the reaction with by the n -th consecutive species in the Bateman chain(22) obtaining:



The second reaction from (43) can be looked as an operator reaction transforming an input species P_n into an output species X . In terms of dynamical reaction equations the ‘‘Gompertzian’’ differential operator

$$x' = kxp$$

acts on a given input function p , producing an output function x . To have an appropriate solution x , we should investigate the properties of the function p such as continuity, boundlessness, sigmoidality, etc..

4 Some other applications of reaction networks:

4.1 Oscillating Chemical Reactions

The Law of Mass Action led to the believe that reactions approach equilibrium steadily. Indeed, most reactions proceed smoothly, at varying rates, to a final state of equilibrium. Some, however, do not. They oscillate in time: reactant, product, or intermediate species' concentrations fluctuate wildly, often leading to easily observable oscillations in time of these concentrations. The Lotka-Volterra model is the simplest mathematical model which exhibits such oscillations. Since natural biological systems have memory properties, fractional differential equations provide an excellent instrument in this respect in comparison to the classical integer-order counterparts [3]. For example, for the Lotka-Volterra model, it is revealed that the so-called

memory concept of the fractional derivative damps out oscillations in the population numbers so that the system as a whole settles on an equilibrium quicker than it would with integer time derivatives. We will focus our attention on other two oscillation reactions - the brusselator and the oregonator which also could be represented with fractional derivatives models.

Brusselator. The Brusselator model is a famous model of chemical reactions with oscillations. In the middle of the last century, Belousov and Zhabotinsky discovered chemical systems exhibiting oscillations. While for most chemical reactions a state of homogeneity and equilibrium is quickly reached, this reaction is a remarkable chemical reaction that maintains a prolonged state of non-equilibrium leading to macroscopic temporal oscillations and spatial pattern formation that is very life-like. After the discovery of oscillating chemical reactions, in 1968 Prigogine proposed a virtual oscillating chemical reaction system - the Brusselator model. This dynamical system holds a pivotal role in the study of chemical kinetics, or biochemical reactions, and biological systems. The dynamical Brusselator reaction-diffusion system involves a controlled concentration of paired variables intermediates with reactants and product chemicals with nonlinear oscillations. It is characterized by the reactions:



where A and B are reactants, D and E are products. It consists of four reactions(44), where the second one is autocatalytic, which we studied as a private case in Generalized Verhulst Differential Model in (19). Indeed, it is the autocatalytic reaction that causes the chemical oscillations in the Brusselator model. It also arises in the formation of ozone by atomic oxygen via a triple collision, in enzymatic reactions, and in plasma and laser physics in multiple couplings between

modes. If we note the concentration of reactants A, B, D, E, X, Y as a, b, d, e, u, v , we can look at every reaction as a step of the brusselator reaction network and write the following differential equations for every step [8] :

Step	Reaction	Differential Equations
1	$A \xrightarrow{k_1} X$	$u' := k_1 a, a' := -k_1 u$
2	$2X + Y \xrightarrow{k_2} 3X$	$u' := k_2 u^2 v, v' := -k_2 u^2 v$
3	$B + X \xrightarrow{k_3} Y + D$	$v' := k_3 u b, d' := k_3 u b, b' := -k_3 b u, u' = -k_3 b$
4	$X \xrightarrow{k_4} E$	$e' := k_4 u, u' := -k_4 u$

Table 1: The steps of the Brusselator reaction network with the equations implied by them

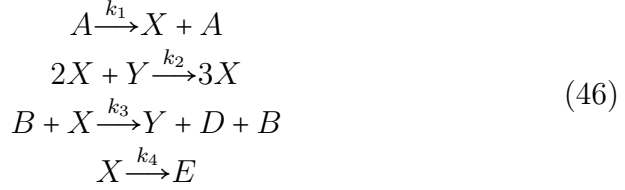
After summarising the single steps from Table 1 , the total dynamics of the Brusselator reaction can be described by a system of two ODEs. In dimensionless form, they are:

$$\begin{cases} u' = k_1 a + k_2 u^2 v - k_3 b u - k_4 u \\ v' = k_3 b u - k_2 u^2 v \end{cases}$$

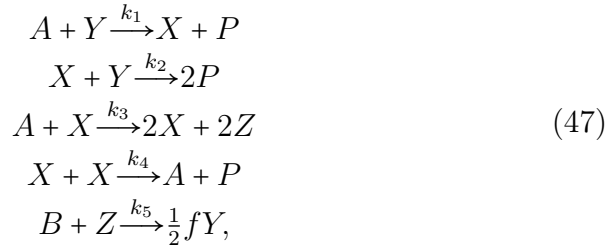
The rate constants k_1 and k_2 are superfluous since the rate of reactions for which they are rate constants can be varied by changing the parameters B and A. Similarly, the rate constant k_3 of the autocatalytic step can be made unity by scaling time. Following Tyson [16], the constant k_4 is given the value unity. The model could be generalized by using fractional derivatives equations and assuming that the concentrations A and B are held constant during the chemical reaction:

$$\begin{aligned} \frac{\partial^\alpha u}{\partial t^\alpha} &= a + u^2 v - (b + 1)u \\ \frac{\partial^\beta v}{\partial t^\beta} &= b u - u^2 v \end{aligned} \quad (45)$$

If we assume that A and B are constant, we can use the idea from [17] and then the reactions will look like:



Oregonator. Another theoretical model of the chemical dynamics of the oscillatory Belousov-Zhabotinsky reaction, proposed later, is the Oregonator. The Oregonator dynamics is an activator/inhibitor system containing both an autocatalytic step and a delayed negative feedback loop. It is a reduced model of Belousov-Zhabotinsky reaction, containing five reactions:



where A and B are reactants; X, Y and Z are intermediate species and P is a product, f is an adjustable stoichiometric factor. If we note the concentration of reactants A, B, D, E, X, Y as a, b, d, e, u, v , we can look at the steps of the reaction network again:

Step	Reaction	Differential Equations
1	$A + Y \xrightarrow{k_1} X + P$	$u := k_1av, p' := k_1av, a' := -k_1av, v' := -k_1av$
2	$X + Y \xrightarrow{k_2} 2P$	$p' := k_2uv, u' := -k_2uv, v' := -k_2uv$
3	$A + X \xrightarrow{k_3} 2X + 2Z$	$u' := k_3ua, w' := k_3ua, a' := -k_3ua$
4	$X + X \xrightarrow{k_4} A + P$	$a' := k_4x^2, p' := k_4x^2, u' := -k_4x^2,$
5	$B + Z \xrightarrow{k_5} \frac{1}{2}fY$	$v' := k_5bw, b' := -k_5bw, w' := -k_5bw$

Table 2: The steps of the Oregonator reaction network with the equations implied by them

For modeling the Oregonator Mass-Action dynamics, we can use again fractional-derivatives equations:

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} &= k_1 av - k_2 uv + k_3 ua - 2k_4 u^2 \\ \frac{\partial^\beta v}{\partial t^\beta} &= -k_1 av - k_2 uv + \frac{1}{2}fbw \\ \frac{\partial^\gamma w}{\partial t^\gamma} &= 2k_3 au - k_5 bw\end{aligned}\tag{48}$$

5 Conclusion

Mathematical modeling of chemical reaction networks consists of a variety of methods for approaching questions about the dynamical behavior of chemical reactions arising in real-world applications.[7] In this paper, we focused our attention on studying the behavior of reactants and products of autocatalytic reactions. The main tool we used was the Law of Mass Action, see section 2.1. The reaction network approach clearly explains the close links between the Gompertz model and the Verhulst logistic model [9]. The new look at the Gompertz-Bateman model [10] as a model of two interacting species U and X, where the dynamics of the catalyst species U is separately determined and independent from that of the growing species X can be further developed by looking for various possibilities for the dynamics of the catalyst species. The proposed reaction network realization of the Gompertz growth model can be interpreted from the perspective of demographic and socioeconomics sciences. Also, the sigmoidal Verhulst and Gompertzian type functions find numerous applications in fitting real-life biological, ecological, socioeconomics, etc. experimental measurement data. The current work proposes some original and innovative ideas expanding the existing ones and adding a new vision of some of the concepts. The adding of the reactant as a product of the reaction in the Verhulst Logistic Differential Model [17] doesn't change the known results from [11] but gives a more natural and logical expression of the idea that the concentration of the catalyst remains constant through the reaction. This is a new kind of approach which is not known in the existing studies of the logistic model. The exploration of the reversible reaction of the Verhulst Logistic Differential

Model with all the possible cases for the rate constants has not been encountered in any other study and gives finished look for the model. The Verhulst Differential Model is generalized as a reaction of multiple reactants and products. The Gompertz-Bateman Growth/Decay model is examined [10] and extended with the reversible reaction and it is found a relationship between the concentrations of the species and their initial conditions. 36

Additionally, there are shown some applications of the proposed ideas in other types of reactions and events.

Some of the presented reaction networks (such as the logistic type ones and the exponential radioactive decay) induce as solutions growth-decay functions possessing exact algebraic solutions. Based on such solutions some authors develop various extensions and modifications obtaining thus various classes of functions that are suitable for approximating measurement data coming from biological and eco-epidemiological processes and phenomena. [12] [13]

6 Acknowledgments

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References

- [1] Anguelov R, Borisov M, Iliev A, Kyurkchiev N, Markov S. On the chemical meaning of some growth models possessing Gompertzian-type property. *Math Meth Appl Sci.* 2018;41:8365–8376. <https://doi.org/10.1002/mma.4539>.
- [2] Bateman H., The solution of a system of differential equations occurring in the theory of radio-active transformations, *Proc. Cambridge Phil. Soc.*, **15** (1910), 423–427.
- [3] Bazhlekova E. and Tsocheva K., Fractional Burgers' model: thermodynamic constraints and completely monotonic relaxation function. *C. R. Acad. Bulg. Sci.*, 69(7) 2016, 825–834.
- [4] Chellaboina, V., S. P. Bhat, W. M. Haddad, D. S. Bernstein, Modeling and Analysis of Mass-Action Kinetics, *IEEE Control Systems Magazine*, August 2009, 60–78.
- [5] Feinberg, M., Foundations of Chemical Reaction Network Theory, *Applied Mathematical Sciences* 202(2019), Springer, <https://doi.org/10.1007/978-3-030-03858-8>
- [6] Gerlee, P. (2013). The Model Muddle: In Search of Tumor Growth Laws. *Cancer research.* 73. 10.1158/0008-5472.CAN-12-4355.
- [7] Goriely A., *The Mathematics and Mechanics of Biological Growth*, Springer New York, 2017
- [8] Kang, H. and Pesin, Y. Milan Dynamics of a Discrete Brusselator Model: Escape to Infinity and Julia Set *j. math.* (2005) 73: 1. <https://doi.org/10.1007/s00032-005-0036-y>
- [9] Markov, S., Reaction networks reveal new links between Gompertz and Verhulst growth functions, *Biomath* 8 (2019), 1904167, <http://dx.doi.org/10.11145/j.biomath.2019.04.167> *Math Meth Appl Sci.* 2017;1-12 <https://doi.org/10.1002/mma.4539>

- [10] Markov S., On a Class of Generalized Gompertz-Bateman Growth-decay Models. *Biomath Communications* 2019, Vol 6, No 1 <http://dx.doi.org/10.11145/bmc.2019.07.307>
- [11] Murray, J.D., 2002. *Mathematical Biology I: An Introduction*, 3rd Ed. Springer, New York.
- [12] N. Kyurkchiev, On a sigmoidal growth function generated by reaction networks. Some extensions and applications, *Communications in Applied Analysis*, 23, 3, 2019, ISSN:1083-2564, 383–400
- [13] N. Kyurkchiev, V. Kyurkchiev, A. Iliev, A. Rahnev, Some non-standard differential models with applications to the population dynamics and computer viruses propagation, *Dynamic Systems and Applications*, 28, No. 3 (2019), 757–788
- [14] Radchenkova N., M. Kambourova, S.Vassilev, R. Alt, S. Markov, On the Mathematical Modelling of EPS Production by a Thermophilic Bacterium, *Biomath* 4 (2014), 1407121, <http://dx.doi.org/10.11145/j.biomath.2014.07.121>
- [15] Savageau, MA., Voit EO., Recasting nonlinear differential equations as Systems: a canonical nonlinear form. *Math Biosci.* 1987;87:83–115.
- [16] Tyson, J.J., Othmer, H.G., 1978. The dynamics of feedback control circuits in biochemical pathways. *Prog. Theor. Biol.* 5, 1–62.
- [17] Verhulst PF., Notice sur la loi que la population poursuit dans son ac- croissance. *Corresp Math Phys.* 1838;10:113–121.

Appendices

A Basic Concepts

Gedankenexperiment - term used by the physicist Albert Einstein to describe his unique approach of using conceptual rather than actual experiments in creating the theory of relativity.

Bifurcation - an event which occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden “qualitative” or topological change in its behavior.

Picard-Lindelöf Theorem Let us consider the initial value problem

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0$$

Suppose f is uniformly Lipschitz continuous in y and continuous in t , then for some value $\epsilon > 0$, there exists a unique solution $y(t)$ to the initial value problem on the interval $[t_0 - \epsilon, t_0 + \epsilon]$.

B Modelling dynamic processes. The case study of the invention of the Free-Fall Law

As a thought experiment ("Gedankenexperiment") let us assume that the Free-Fall Law is not yet invented (or fully accepted) and we need to establish (re-invent) the Free-Fall Law equations by means of contemporary mathematical modelling tools.

Free-Fall Law:

Free fall is any motion of a body where gravity is the only force acting upon it. Then the acceleration is always downward and has the same magnitude for all bodies, commonly denoted g .

For checking the validity of the Free-Fall Law, we use the experimental data. We measure the time at which the body falls to the ground at fixed heights. Then, we want to find the curve on which the points lie, which describes the motion. The ancient Greek philosopher Aristotle (384 - 322 BC) discussed falling objects in what was perhaps the first book on Mechanics. He assumed that all objects fall with a constant speed and that heavy objects fall faster than light. Graphically, Aristotle's model looks as shown on Figure 27. The Italian scientist Galileo Galilei (1564 - 1642) subjected the Aristotelian theories to experimentation and careful observation. He then combined the results of these experiments with mathematical analysis in an unprecedented way. His famous experiment with dropping little balls from Leaning Tower of Pisa are a real example for the principle of mathematical modelling.

First, it is assumed that the bodies fall straight. Using some already found data, this situation will look like this

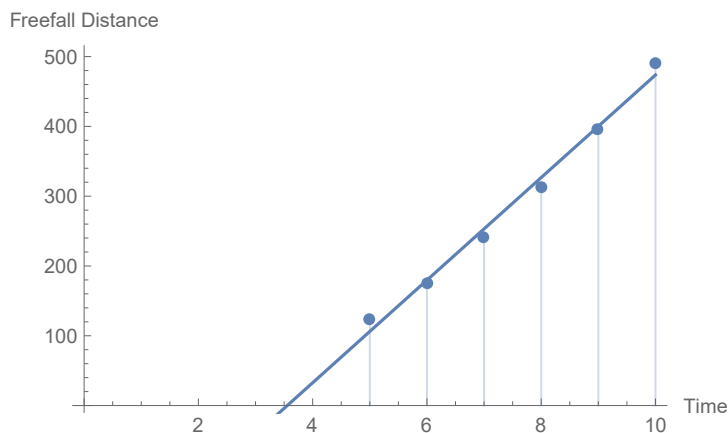


Figure 20: Aristotel's model of Free-Fall Law

However, when we try to fit the data with the parabola, the result is almost exact.

Time	Distance
5	122.5
6	176.4
7	240.1
8	313.6
9	396.9
10	490.0

We can see that, in fact, the points lie on a parabola as shown on Figure 28:

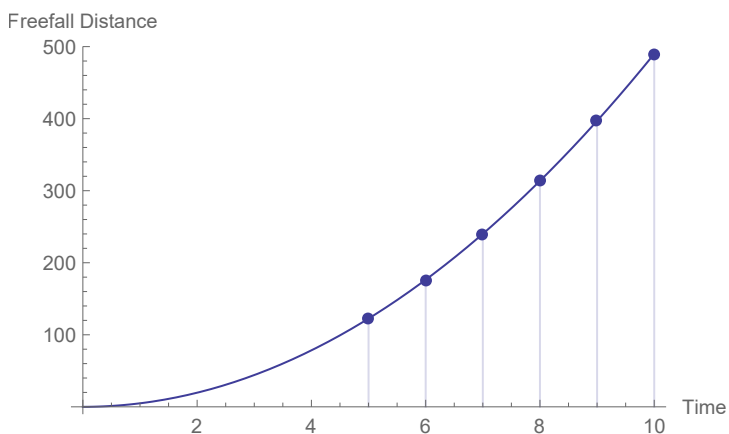


Figure 21: Free-fall experimental data, visualized by parabola

We have seen an example of mathematical modelling. We use this principle when exploring chemical reaction networks.

C The Laplace Integral Transform

The Laplace transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F(s)$, which is a unilateral transform defined by

$$F(s) = \int_0^{+\infty} f(t)e^{-st} dt$$

The Laplace transform is a powerful integral transform used to switch a function from the time domain to the s-domain. The Laplace transform can be used in some cases to solve linear differential equations with given initial conditions. Its usefulness is in stepping down from a more difficult problem to an easier one, for example stepping down from a derivative $\frac{dF}{dt}$ to a function $f(s)$.

$$L\left\{\frac{dF}{dt}\right\} = \int_0^{+\infty} \frac{dF}{dt} e^{-st} dt$$

Integrating by parts, we have:

$$\begin{aligned} L\left\{\frac{dF}{dt}\right\} &= F(t)e^{-st} \Big|_0^{+\infty} - \int_0^{+\infty} F(t)d(e^{-st}) \\ &= -F(t=0) + s \int_0^{+\infty} F(t)(e^{-st})dt \\ &= -F(t=0) + sf(s) \end{aligned}$$

D Application in Calculating p(t)

We substitute $u(t)$ with the solution from $u(t) = u_0 e^{-k_1 t}$ and apply Laplace transform L of both sides:

$$\begin{aligned} \frac{dp(t)}{dt} &= k_1 u_0 e^{-k_1 t} - k_2 p(t) \\ L\left\{\frac{dp(t)}{dt}\right\} &= L\{k_1 u_0 e^{-k_1 t} - k_2 p(t)\} \\ sq(s) - p(t) &= k_1 u_0 \frac{1}{s + k_1} - k_2 q(s) \end{aligned}$$

We assume that $p(t = 0) = p_0$

$$(s + k_2)q(s) = \frac{k_1 u_0}{s + k_1} + p_0$$

$$q(s) = k_1 u_0 \frac{1}{(s + k_2)(s + k_1)} + \frac{p_0}{s + k_2}$$

$$\frac{1}{(s + k_2)(s + k_1)} = \frac{1}{k_2 - k_1} \frac{1}{s + k_1} - \frac{1}{k_2 - k_1} \frac{1}{s + k_2}$$

$$\implies q(s) = \frac{k_1 u_0}{k_2 - k_1} \left(\frac{1}{s + k_1} - \frac{1}{s + k_2} \right) + \frac{p_0}{s + k_2}$$

We apply Inverse Laplace transform:

$$p(t) = \frac{k_1 u_0}{k_2 - k_1} (e^{-k_1 t} - e^{-k_2 t}) + p_0 e^{-k_2 t}$$

The calculation could be made easy using CAS Mathematica:

```

In[23]:= eqn = p'[t] == k1*u0*E^(-k1*t) - k2*p[t];
leqn = ApplySides[LaplaceTransform[#, t, s] &, eqn];
lsol = Solve[leqn, LaplaceTransform[p[t], t, s]];
ilsol = InverseLaplaceTransform[LaplaceTransform[p[t], t, s] /. lsol[[1]], s, t] // FullSimplify;
s1 = p[t] /. {dsol[[1, 1]]} /. {p[0] -> p0}


$$\frac{e^{-k_2 t} (-k_1 p_0 + k_2 p_0 - k_1 u_0 + e^{(-k_1 + k_2) t} k_1 u_0)}{-k_1 + k_2}$$


```

Figure 22: Laplace transform for finding the solution for p