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The dynamics and control of a singular biological economic model with stage structure. Some moduli in programming environment Mathematica[1]

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Abstract

Proposed are some modifications of bio-economic models with stage structure [3]–[8]. The proposed modifications aim at a better description and understanding of the underlying bio-economic mechanisms. This is achieved by formulating the model in terms of chemical-type reaction steps. Several new modules in the Computer Algebra System *MATHEMATICA* are proposed, offering a possibility for visualization of the resulting solutions and sensitive analysis of the model.

Keywords. Biological economic model with stage structure, Environmental pollution factors, Capture capability of mature population, Economic profit, Programming environment *MATHEMATICA*.

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1 Introduction

The following single population model involving mature and immature population is proposed and investigated in ([4], [7], [8]):

$$(A)\begin{cases} x'(t) &= ay(t) - bx(t) - r_1 x(t) \\ y'(t) &= bx(t) - r_2 y(t) - \beta y^2(t), \end{cases}$$

where $x(t)$ and $y(t)$ are the densities of the immature population and mature population at time t, resp., a denotes the birth rate of the immature population, r_1, r_2 are the death rates of the immature population and mature population, resp., b denotes the conversion rate from immature population into mature population, β denotes the intra-specific effect coefficient.

A single population model in a polluted environment is investigated in [5]:

$$(B)\begin{cases} x'(t) &= rx(t)\left(1 - \dfrac{x(t)}{K}\right) - r_1 u(t)x(t) \\ u'(t) &= \theta - hu(t), \end{cases}$$

where $x(t)$ is the population density, $u(t)$ is the concentration of environment pollutants, r denotes the intrinsic growth rate when there is no pollution, K denotes the capacity of the environment, $r_1 u(t)$ can be interpreted as the measuring response function of the reduction of populations because of the pollution factor, θ denotes the amount of pollutants coming from outside, $hu(t)$ can be interpreted as the reduction of pollutant concentration due to other factors.

Based on system (A) and (B), a single population model with stage structure is given in [5]:

$$\begin{aligned} x'(t) &= ay(t) - bx(t) - r_1 x(t) - \eta_1 u(t)x(t) \\ y'(t) &= bx(t) - r_2 y(t) - \beta y^2(t) - \eta_2 u(t)y(t) \\ u'(t) &= \theta - hu^2(t), \end{aligned} \tag{1}$$

where $\eta_1 u(t)$, $\eta_2 u(t)$ can be interpreted as the measuring response function of the reduction of population because of the pollution factor, $hu^2(t)$ can be interpreted as the reduction pollutant concentration because of other factors.

A singular biological economic model with environmental pollution factors is proposed as follow [3], [6]:

$$\begin{aligned} x'(t) &= ay(t) - bx(t) - r_1x(t) - \eta_1 u(t)x(t) \\ y'(t) &= bx(t) - r_2y(t) - \beta y^2(t) - \xi E(t)y(t) - \eta_2 u(t)y(t) \\ u'(t) &= \theta - hu^2(t) \end{aligned} \tag{2}$$

$$0 = E(t)\,(py(t) - c) - m, \tag{3}$$

where

- $E(t)$ is the capture capability of mature population at the time t;
- p denotes the unit price;
- c denotes the unit cost;
- m denotes the economic profit;
- $pE(t)y(t)$ is the total revenue;
- $cE(t)$ is the total cost.

Problem. Find function $E(t)$ satisfying equation (3) in the context of model (2).

We study here the structure of the capture capability $E(t)$ of the form:

$$E(t) = \frac{lt^2}{1 + dt^2}$$

(the parameters l and d are positive).

Then the above system (2)–(3) is written as follows:

$$\begin{aligned} x'(t) &= ay(t) - bx(t) - r_1x(t) - \eta_1 u(t)x(t) \\ y'(t) &= bx(t) - r_2y(t) - \beta y^2(t) - \xi \frac{lt^2}{1 + dt^2} y(t) - \eta_2 u(t)y(t) \\ u'(t) &= \theta - hu^2(t) \end{aligned} \tag{4}$$

$$0 = \frac{lt^2}{1+dt^2}\left(py(t)-c\right) - m. \tag{5}$$

2 Implementations in programming environment *MATHEMATICA*

Modulus I. The model (1). The code and the test provided on our control examples are shown in Fig. 1 and Fig. 2.

Modulus II. The model (4)-(5). The code is shown in Fig. 3.

Specifically, we note that the task of minimizing the functional of several variables (derived from the model problem (2)-(3)) is correctly studied in [3]–[8].

The system (4) - (5) can be written as:

$$A(t)X'(t) = G\left(x(t), y(t), u(t), \frac{lt^2}{1+dt^2}\right),$$

where

$$X(t) = \left(x(t), y(t), u(t), \frac{lt^2}{1+dt^2}\right),$$

```
Print["Modulus I. The model (1)"];

a = Input["Input -  a"]; (* 1 *)
Print[" a = ", a];
b = Input["Input -  b"]; (* 1 *)
Print[" b = ", b];
r1 = Input["Input -  r1"]; (* 0.5 *)
Print[" r1 = ", r1];
η1 = Input["Input -  η1"]; (* 0.1 *)
Print[" η1 = ", η1];
r2 = Input["Input -  r2"]; (* 0.4 *)
Print[" r2 = ", r2];
β = Input["Input  -  β"]; (* 0.2 *)
Print[" β = ", β];
η2 = Input["Input -  η2"]; (* 0.15 *)
Print[" η2 = ", η2];
θ = Input["Input  -  θ"]; (* 1 *)
Print[" θ = ", θ];
h = Input["Input -  h"];(* 0.6 *)
Print[" h = ", h];
x0 = Input["Input initial condition - x[0]"]; (* 1 *)
Print["Initial condition x0 = ", x0];
y0 = Input["Input initial condition - y[0]"]; (* 0.1 *)
Print["Initial condition y0 = ", y0];
u0 = Input["Input initial condition - u[0]"]; (* 1 *)
Print["Initial condition u0 = ", u0];
t0 = Input["Input t0"];
Print["t0 = ", t0];
t1 = Input["Input t1"];
Print["t1 = ", t1];

Print["Graphics of the solutions of the system of differential equations
as functions of the time t"];
NDSolve[{x'[t] == a*y[t] - b*x[t] - r1*x[t] - η1*u[t]*x[t],
   y'[t] == b*x[t] - r2*y[t] - β*(y[t])^2 - η2*u[t]*y[t], u'[t] == θ - h*(u[t])^2,
   x[0] == x0, y[0] == y0, u[0] == u0}, {x, y, u}, {t, t0, t1}];

Plot[Evaluate[{x[t], y[t], u[t]} /. First[%]], {t, t0, t1}]
exactsol = NDSolve[{x'[t] == a*y[t] - b*x[t] - r1*x[t] - η1*u[t]*x[t],
     y'[t] == b*x[t] - r2*y[t] - β*(y[t])^2 - η2*u[t]*y[t], u'[t] == θ - h*(u[t])^2,
     x[0] == x0, y[0] == y0, u[0] == u0}, {x, y, u}, {t, t0, t1}];
```

Figure 1: Modulus in *CAS Mathematica*

Modulus I. The model (1)

$a = 1$
$b = 1$
$r1 = 0.5$
$\eta 1 = 0.1$
$r2 = 0.4$
$\beta = 0.2$
$\eta 2 = 0.15$
$\theta = 1$
$h = 0.6$
Initial condition $x0 = 1$
Initial condition $y0 = 0.1$
Initial condition $u0 = 1$
$t0 = 0$
$t1 = 20$

Graphics of the solutions of the system of differential equations
as functions of the time t

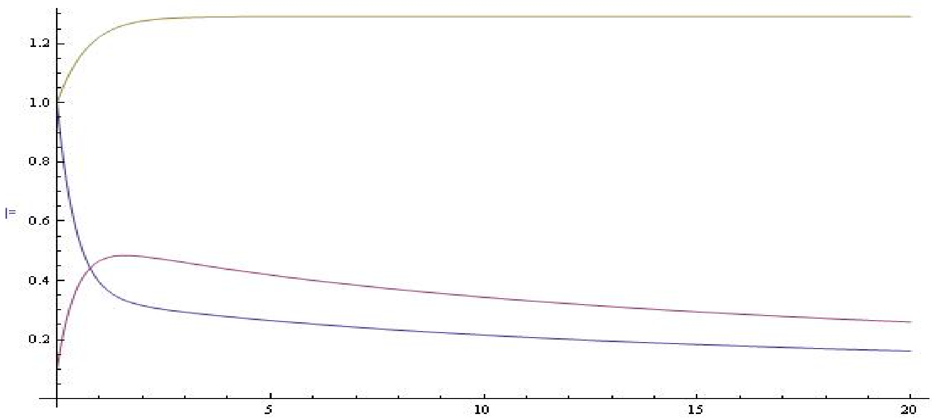


Figure 2: The test provided on our control examples

$$A(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$G\left(x(t), y(t), u(t), \frac{lt^2}{1+dt^2}\right)$$
$$= \begin{pmatrix} ay(t) - bx(t) - r_1x(t) - \eta_1u(t)x(t) \\ bx(t) - r_2y(t) - \beta y^2(t) - \xi\frac{lt^2}{1+dt^2}y(t) - \eta_2u(t)y(t) \\ \theta - hu^2(t) \\ \frac{lt^2}{1+dt^2}\left(py(t) - c\right) - m \end{pmatrix}.$$

The proposed modified model (4)–(5) gives possibility for approximation of the function $E(t)$ with fractional rational function and conducting ultra-sensitive analysis through the build in software tools for additional animations.

3 Conclusion remarks

If the asymptotic behavior of the function $E(t)$ at $t \to \infty$ is known, then it is not difficult to take into account that from the model problem (4)–(5) we have

$$\lim_{t\to\infty} \frac{lt^2}{1+dt^2} = \frac{l}{d},$$

which allows the variation of the free parameters l and d.

The last part of the presented program code contains some "software evidence" that equation (5) is satisfied.

Various related nonlinear models are discussed in [1], [2].

```
Print["The model (4)-(5)"];

Print["The Modelling and Control of a Singular Biological Economic System in a Polluted Environment "];

Print["A singular biological economic model with environmental pollution factors is proposed as follows: "];

Print["x'[t] ==a*y[t]-b*x[t]-r1*x[t]-η1*u[t]*x[t]"];
Print["y'[t] ==b*x[t]-r2*y[t]-β*(y[t])^2-ξ*E[t]*y[t]-η2*u[t]*y[t]"];
Print["u'[t]==θ-h*(u[t])^2"];
Print["E[t](p*y[t]-c)-m==0,"];

Print["where E[t] is the capture capability of mature population at the time t,"];
Print["p denotes the unit price,"];
Print["c denotes the unit cost,"];
Print["m denotes the economic profit,"];
Print["p*E[t]*y[t] is the total revenue,"];
Print["c*E[t] is the total cost."];

Print["We study here the structure of the capture capability E[t] of the form:"];
```

$$\text{Print}\left[\text{"E[t]==}\frac{1*t\hat{}2}{1+d*t\hat{}2}\text{. "}\right];$$

```
Print["Then, the above system is written as the following:"];

Print["x'[t] ==a*y[t]-b*x[t]-r1*x[t]-η1*u[t]*x[t]"];
```

$$\text{Print}\left[\text{"y'[t] ==b*x[t]-r2*y[t]-}\beta\text{*(y[t])\^{}2-}\xi\text{*}\frac{1*t\hat{}2}{1+d*t\hat{}2}\text{*y[t]-}\eta\text{2*u[t]*y[t]"}\right];$$

```
Print["u'[t]==θ-h*(u[t])^2"];
```

$$\text{Print}\left[\text{"}\frac{1*t\hat{}2}{1+d*t\hat{}2}\text{(p*y[t]-c)-m==0."}\right];$$

```
a = Input["Input -  a"];(* 2 *)
Print[" a = ", a];
b = Input["Input -  b"];(* 1 *)
Print[" b = ", b];
r1 = Input["Input -  r1"];(* 0.2 *)
Print[" r1 = ", r1];
η1 = Input["Input -  η1"];(* 0.3 *)
Print[" η1 = ", η1];
r2 = Input["Input -  r2"];(* 0.3 *)
Print[" r2 = ", r2];
β = Input["Input  -  β"];(* 0.2 *)
Print[" β = ", β];
ξ = Input["Input -  ξ"];(* 0.5 *)
Print[" ξ = ", ξ];
η2 = Input["Input -  η2"];(* 0.2 *)
Print[" η2 = ", η2];
θ = Input["Input  -  θ"];(* 1 *)
Print[" θ = ", θ];
h = Input["Input -  h"];(* 1 *)
Print[" h = ", h];
p = Input["Input -  p"];(* 1 *)
Print[" p = ", p];
c = Input["Input -  c"];(* 1 *)
Print[" c = ", c];
l = Input["Input -  l"];(* 4 *)
Print[" l = ", l];
d = Input["Input -  d"];(* 5 *)
Print[" d = ", d];

x0 = Input["Input initial condition - x[0]"]; (* 1.3 *)
Print["Initial condition x0 = ", x0];
y0 = Input["Input initial condition - y[0]"]; (* 0.6 *)
Print["Initial condition y0 = ", y0];
u0 = Input["Input initial condition - u[0]"]; (* 0 *)
Print["Initial condition u0 = ", u0];
t0 = Input["Input t0"];
Print["t0 = ", t0];
t1 = Input["Input t1"];
Print["t1 = ", t1];
```

```
eqns = {
   x'[t] == a*y[t] - b*x[t] - r1*x[t] - η1*u[t]*x[t],
   y'[t] == b*x[t] - r2*y[t] - β*(y[t])^2 - ξ*(l*t^2)/(1+d*t^2)*y[t] - η2*u[t]*y[t],
   u'[t] == θ - h*(u[t])^2};
Print["Graphics of the solutions of the system of differential equations as functions of the time t:"];

NDSolve[{eqns, x[0] == x0, y[0] == y0, u[0] == u0}, {x, y, u}, {t, t0, t1}];

Plot[{Evaluate[{x[t], y[t], u[t]} /. First[%]], (l*t^2)/(1+d*t^2)}, {t, t0, t1}]
exactsol = NDSolve[{eqns, x[0] == x0, y[0] == y0, u[0] == u0}, {x, y, u}, {t, t0, t1}];

data = Table[y[t] /. exactsol[[1]], {t, t0 + 1, t1}];
ListPlot[data, PlotRange → {0, 3}, PlotLabel → y[t], LabelStyle → Directive[Green, Bold], PlotStyle → Orange]


Manipulate[Plot[(l*t^2)/(1+d*t^2), {t, t0 + 1, t1}], {l, 1, 10, Appearance → "Open"}, {d, 1, 20, Appearance → "Open"},
 PlotRange → {0, 10}]

Print["The optimal controller m=0.92"];

Print["Proof."];

data1 = Table[c / p + 0.92*(1 + d*t^2) / (l*t^2), {t, t0 + 1, t1}];
ListPlot[data1, PlotRange → {0, 3}, PlotLabel → c / p + 0.92*(1 + d*t^2) / (l*t^2),
 LabelStyle → Directive[Green, Bold], PlotStyle → Orange]

ListPlot[{data, data1}, PlotMarkers → Automatic, PlotRange → {0, 3}]

Plot[(l*t^2)/(1+d*t^2), {t, 1, 250}, PlotLabel → (l*t^2)/(1+d*t^2), LabelStyle → Directive[Green, Bold], PlotStyle → Red]

data2 = data - data1;
ListPlot[data2, PlotRange → {0, 3}, PlotLabel → y - (c / p + 0.92*(1 + d*t^2) / (l*t^2)),
 LabelStyle → Directive[Green, Bold], PlotStyle → Orange]
```

The Modelling and Control of a Singular Biological Economic System in a Polluted Environment

A singular biological economic model with environmental pollution factors is proposed as follows:

x'[t] ==a*y[t]-b*x[t]-r1*x[t]-η1*u[t]*x[t]

y'[t] ==b*x[t]-r2*y[t]-β*(y[t])^2-ξ*E[t]*y[t]-η2*u[t]*y[t]

u'[t]==θ-h*(u[t])^2

E[t](p*y[t]-c)-m==0,

where E[t] is the capture capability of mature population at the time t,

p denotes the unit price,

c denotes the unit cost,

m denotes the economic profit,

p*E[t]*y[t] is the total revenue,

c*E[t] is the total cost.

We study here the structure of the capture capability E[t] of the form:

$$E[t]==\frac{l*t^2}{1+d*t^2}.$$

Then, the above system is written as the following:

x'[t] ==a*y[t]-b*x[t]-r1*x[t]-η1*u[t]*x[t]

$$y'[t] ==b*x[t]-r2*y[t]-\beta*(y[t])^2-\xi*\frac{l*t^2}{1+d*t^2}*y[t]-\eta 2*u[t]*y[t]$$

u'[t]==θ-h*(u[t])^2

$$\frac{l*t^2}{1+d*t^2}(p*y[t]-c)-m==0.$$

a = 2

b = 1

r1 = 0.2

η1 = 0.3

r2 = 0.3

β = 0.2

ξ = 0.5

η2 = 0.2

θ = 1

h = 1

p = 1

c = 1

l = 4

d = 5

Initial condition x0 = 1.3

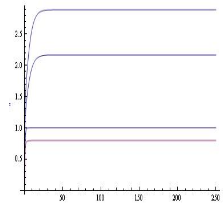
Initial condition y0 = 0.6

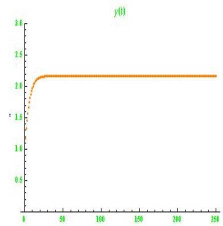
Initial condition u0 = 0

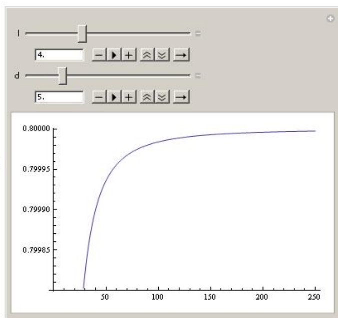
t0 = 0

t1 = 250

Graphics of the solutions of the system of differential equations as functions of the time t:

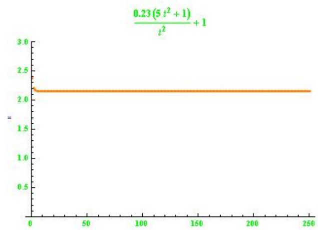






The optimal controller m=0.92

Proof.



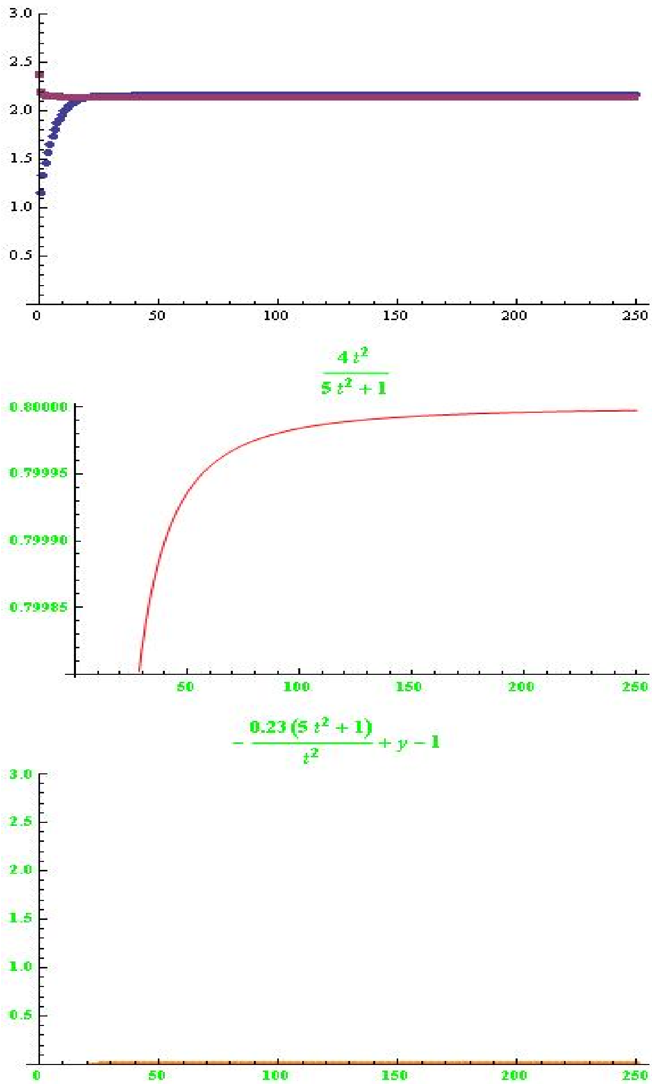


Figure 3: Modulus in *CAS Mathematica* for visualization of the resulting solutions and "software evidence" to satisfying Eq. (5).

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