



# Model phytoplankton population by branching processes

Antoanela Terzieva

Sofia University "St. Kliment Ohridski", Bulgaria

Antoanela.t@abv.bg

## Abstract

The purpose of this research is to model the dynamics of phytoplankton population and chlorophyll-a contained therein, usually modeled by differential equations. As a modelling tool we propose the multitype branching stochastic processes of Bellman-Harris. The proposed models could be applicable not only for populations of unicellular organisms, but also for arbitrary sets of objects which operate a certain period of time and then split into two newborn objects.

## 1 Introduction

It is customary to call phytoplankton the microscopic aquatic species, mostly unicellular. Phytoplankton is one of the most ancient inhabitants of our planet, playing key role at the base of the ocean and marine food chains. It also controls the global carbon cycle which has a significant impact on the climate regulation. Last but not least phytoplankton is a key factor in ecology.

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The purpose of this research is to model the dynamics of phytoplankton population, usually modeled by differential equations [8]. The proposed models could be applicable not only for populations of unicellular organisms, but also for arbitrary sets of objects which operate a certain period of time and then split into two newborn objects.

Data on the amount of chlorophyll-a may be obtained by satellite and the chlorophyll-a quantity in a unit biomass phytoplankton is known [3,4]. Therefore one of our goals is to give a possible model, such that, wherever in we can treated as a single particle not the whole cell phytoplankton, only the chlorophyll-a quantity, contained in.

The article consists of three sections. In the next, second section, we try to analyse the difference between the cell division mechanisms and propose as a model multitype branching stochastic process of Bellman-Harris. There are derived equations for the generating functions. Respectively there are expressed the first moments, as well as their particular cases are presented in the second part of the second section. In Section 2.2. also are obtained the expected number of particles. There are estimated the asymptotic behaviour in the infinity for the obtained three-types processes. In the third section it is presented the model simulation of the diatom population.

## 2 Phytoplankton model

The cells reproduce by cell division, split into two daughter cells, in general not identical. In this process the role of the particle will be played by a single phytoplankton cell. Assume particles evolve independently (branching condition). There are about six hundred different species phytoplankton on the Bulgarian Black Sea coast. From now on, we focus on one of the species and take in consideration arbitrary time. (For example, diatoms, one of the most common marine species phytoplankton, their size may vary between thirty and a hundred and fifty microns [5].)

Let  $r = r(T)$  be the birth size of a particle  $T$ . More accurately  $r$  is the maximal distance between two points on particle  $T$ . Let  $A^i(x^i, y^i, z^i)$  and  $A^j(x^j, y^j, z^j)$  be two random points over the particle.

That is to say

$$r = \max_{i,j} \sqrt{(x^i - x^j)^2 + (y^i - y^j)^2 + (z^i - z^j)^2}. \quad (1)$$

It is assumed an additional structure of the birth size motion in line. We consider a parent particle denoted by  $\Gamma_0$ , and assume a birth size  $r_0$ . After a random time  $\Gamma_0$  dies or splits into two particles  $\Gamma_{01}$  and  $\Gamma_{02}$ , which birth sizes then move to the random points  $r_{01} = r_0 + x_1$ ,  $r_{02} = r_0 + x_2$ . Let us designate the birth size-shift by  $\Delta\lambda$ . For any variety phytoplankton then exists min and max  $r(T) := d''$ . Divide  $\Delta := [d', d'']$  into  $n$  subintervals  $\Delta_1, \dots, \Delta_n$  with lengths  $a := (d'' - d')/n$ , endpoints  $[c_i, c_{i+1}]$  and midpoints  $r_i := (c_i + c_{i+1})/2$ , where  $c_0 = d'$ ,  $c_i := c_0 + ai$ . Designate  $k(i) = \min(i - 1, n - i)$ . The cells are born, evolve and divide with probability  $\rho$  or die without leaving progeny with probability  $(1 - \rho)$ . Let the random variables  $\Lambda r$  and  $\tau_i$  are defined over the probability space  $(\Omega, \mathfrak{F}, P)$  for integer,  $n0$  and  $i = 1, \dots, n$ . Consider the process

$$\{\mathbf{Z}(\mathbf{t}), t \geq 0 \mid \mathbf{Z}(\mathbf{0}) = Z_0\}, \mathbf{Z}(\mathbf{t}) = (Z_1(t), \dots, Z_n(t)). \quad (2)$$

If  $T_i$  reaches maturity separated to  $T_{i+l}$  and  $T_{i-l}$  with probability  $p_{i(i+l)}$ ;  $i = 1, \dots, n$  and  $l = 0, 1, 2, \dots, k(i)$ .  $T_i$  has for a lifetime the random variable (r.v.)  $\tau_i$ , with distribution function (d.f.)  $G_i(t) = P(\tau_i \leq t)$ ,  $i = 1, \dots, n$ . Assume that every cell was born of zero age and exclude the instantaneous death at birth. Therefore  $G_i(0) = G_i(0+) = 0$ .  $\{\mathbf{Z}(\mathbf{t}), t \geq 0\}$  is a multitype branching process of Bellman-Harris (BPBH) [6, 10], it is logical to accept that  $p_{i(i-l)} = p_{i(i+l)}$  for  $l = 0, 1, \dots, k(i)$ . The mechanism of cell division of the particles is illustrated in Figure 1.

Denote sets of points

$$C := \{-k(i), -k(i) + 1, \dots, 0, 1, \dots, k(i)\}, \quad (3)$$

$$D := \{i = 1; j = 2, \dots, n\} \cup \{i = n; j = 1, \dots, (n - 1)\}, \quad (4)$$

and define  $p_{i(i+l)}$  as follows [12]

**Definition 1.**

$$p_{i(i+l)} = \begin{cases} 0 & l \notin C \\ \frac{1}{\sigma_i \sqrt{2\pi}} \int_{c_{i+l}}^{c_{i+l+1}} e^{-\frac{(t-r_i)^2}{2\sigma_i^2}} dt & l \in C \\ 1 - \sum_{\substack{j=1 \\ j \neq i}}^n p_{ij} & l = 0 \quad (\text{i.e. } j = i) \end{cases}$$

Therefore  $p_{11} = p_{nn} = 1$ ;  $\sum_{j=1}^n p_{ij} = 1$ ;  $i = 1, \dots, n$ ;  $p_{ij} = 0$  for  $i, j \in D$ .

For each phytoplankton cell the following conditions are fulfilled

**Conditions 1**

1. Any  $T_i$  reaches maturity with a probability  $\rho$ , when splits into exactly two daughters  $T_{i+l}$  and  $T_{i-l}$ .

2. The birth size  $r$  consists of a latent factor handed down by the mother, and of an individual contribution [7]. The shift of  $r$  is  $\Delta\lambda \in \tilde{N}(0, \sigma^2)$ , where the transition probabilities  $p_{ij}$  are given by Definition 1.

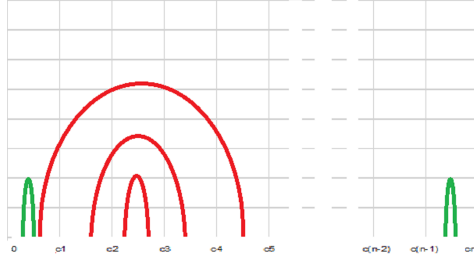


Figure 1: Mechanism of division for particles  $T_1, T_2$  and  $T_n$ .

## 2.1 Equations

Denote  $N_n = \{\alpha = (\alpha_1, \dots, \alpha_n) : \alpha_i = 0, 1, \dots; i = 1, \dots, n\}$   $r = (r_1, \dots, r_n)$ ,  $\vec{0} = \underbrace{(0, \dots, 0)}_n$ ,  $\vec{1} = \underbrace{(1, \dots, 1)}_n$ ,  $s = (s_1, \dots, s_n)$ ,  $F = (F_1, \dots, F_n)$ .

**Proposition 1.** For the moments of the  $n$ -types BPBH  $Z(t)$  with any d.f.  $G_i(t)$  and with probability law (p.l.) as in Definition 1 we have

$$m_{ij}(t) = \delta_{ij}[1 - G_i(t)] + 2\rho p_{ii} \int_0^t m_{ij}(t-u) dG_i(u) \\ + \rho \sum_{l=1}^{k(i)} p_{i(i+l)} \int_0^t [m_{(i-l)j}(t-u) + m_{(i+l)j}(t-u)] dG_i(u). \quad (5)$$

**Proof.** The individual generating function (i.g.f.) of  $\mathbf{Z}(t)$  is  $\mathbf{f}(s) = (f_1(s), \dots, f_n(s))$ . Respectively by components for  $i = 1, \dots, n$  we have

$$f_i(s_1, \dots, s_n) = 1 - \rho + \rho \sum_{l=0}^{k(i)} p_{i,i+l} s_{i-l} s_{i+l}. \quad (6)$$

For  $i = 1, n$  we receive

$$f_i(s_1, s_2, \dots, s_n) = 1 - \rho + \rho s_i^2. \quad (7)$$

For the generating function (g.f.)  $F(t; s) = E\{s^{Z(t)} \mid Z(0) = \vec{1}\}$  it is valid

$$F(t, s) = s[1 - G(t)] + \int_0^t f[F(t-u, s)] dG(u) \\ = \sum_{r \in N_n} P\{Z(t) = r \mid Z(0) = \vec{1}\} s_1^{r_1} \dots s_n^{r_n}. \quad (8)$$

$$F_i(t, s) = E\{s^{Z(t)} \mid Z(0) = e_i\} \\ = \sum_{r \in N_n} P\{Z(t) = r \mid Z(0) = e_i\} s_1^{r_1} \dots s_n^{r_n}. \quad (9)$$

From (6) and (8) we obtain

$$F_i(t, s) = s_i[1 - G_i(t)] + \\ \int_0^t f_i[F_{i-k(i)}(t-u, s), F_{i-k+1}, \dots, F_{i+k}(t-u, s)] dG_i(u). \quad (10)$$

Therefore

$$F_i(t, s) = s_i[1 - G_i(t)] + (1 - \rho)G_i(t) + \rho \int_0^t [p_{ii}F_i^2 + p_{i(i+1)}F_{i-1}F_{i+1}, \dots, p_{i(i+k(i))}F_{i-k(i)}F_{i+k}]dG_i(u). \quad (11)$$

$$m_{ij}(t) = E\{Z_j(t) \mid Z(0) = e_i\} = \frac{\partial F_i}{\partial s_j} \Big|_{s=\bar{1}}. \quad (12)$$

After differentiation in (10)

$$m_{ij}(t) = \delta_{ij}[1 - G_i(t)] + \rho \sum_{l=0}^{k(i)} p_{i(i+l)} \frac{\partial}{\partial s_j} \int_0^t F_{i-l}(t-u, s) F_{i+l}(t-u, s) dG_i(u) \Big|_{s=\bar{1}}. \quad (13)$$

$$m_{ij}(t) = \delta_{ij}[1 - G_i(t)] + \rho \sum_{l=0}^{k(i)} p_{i(i+l)} \int_0^t [m_{(i-l)j}(t-u) + m_{(i+l)j}] dG_i(u)$$

which proves the proposition.  $\square$

For all  $t \geq 0$   $m_{1j}(t) = 0$  if  $j \neq 1$ ,  $m_{nj}(t) = 0$  if  $j \neq n$ . For  $i = 1, n$  receive  $k(i) = 0$ ,  $l \equiv 0$  and

$$m_{ii}(t) = 1 - G_i(t) + 2\rho p_{ii} \int_0^t m_{ii}(t-u) dG_i(u); i = 1, n. \quad (14)$$

(14) are called equations of renewal, their solutions are known [1,2].

From (12) for all  $n$  and  $j = 1, \dots, n$  we receive

$$m_{2j}(t) = 2\rho p_{22} \int_0^t m_{2j}(t-u) dG_2(u) + \rho p_{23} \int_0^t (m_{1j}(t-u) + m_{3j}(t-u)) dG_2(u). \quad (15)$$

Denote the partial derivatives of  $f_i$  by  $m_{ij}$

$$m_{ij} := \frac{\partial f_i}{\partial s_j} \Big|_{s=\bar{1}} = \begin{cases} 0 & j \neq i+l \quad l=0,1,\dots,k(i) \\ \rho p_{ij} & j = i+l \quad l=1,2,\dots,k(i) \\ 2\rho p_{ii} & j = i \end{cases}$$

Recall that when  $2\rho p_{ii}G(0+) < 1$ , as is the case under consideration, equation (14) admits a unique solution, that is bounded on bounded intervals.

Denote the Laplace transform  $\widehat{G}(u) = \int_0^\infty e^{-\alpha u} dG(u)$ . The associated Malthus parameter  $\alpha$  is defined as the root to the equation  $\theta\widehat{G}(u) = 1$ . We will assume, that  $\alpha$  always exists.

**Proposition 2.** *For  $n$ -types  $Z(t)$  with  $G_i(t) \in \text{Exp}(\lambda_i); i = 1, n$ , we have*

$$m_{11}(t) \sim e^{\lambda_1 t} \frac{(2\rho - 1)(\alpha_1 + \lambda_1)^2}{\alpha_1 4\rho^2}, \quad (16)$$

$$m_{nn}(t) \sim e^{\lambda_n t} \frac{(2\rho - 1)(\alpha_n + \lambda_n)^2}{\alpha_n 4\rho^2}. \quad (17)$$

**Proof.** For  $i = 1, n$  we get  $m_{ii}(t)$ , considering BPBH  $V_t$  with one type of particles. Then for  $m(t)$  being the measure  $m(t) := E\xi(t)$ , number  $m_0$ ,  $A(t) := EV_t$ .

$$A(t) = 1 - G(t) + m \int_0^t A(t-u) dG(u). \quad (18)$$

The Malthus parameter governs the asymptotic behavior of the function  $U(t)$ . When  $m \geq 1$ , the equation  $m\tilde{G}(u) = 1$  has a unique real solution  $\alpha \geq 0$ . When  $m < 1$ , a solution may not exist, but if it does it has to be negative. We have [2]

$$A(t) \sim e^{\alpha t} \frac{\int_0^\infty e^{-\alpha u} (1 - G(u)) du}{m \int_0^\infty u e^{-\alpha u} dG(u)}, \quad (19)$$

therefore

$$A(t) \sim e^{\alpha t} \frac{m-1}{am^2 \int_0^\infty ue^{-\alpha u} dG(u)}. \quad (20)$$

From  $p_{11} = p_{nn} = 1$ , and from the above formulas it follows  $\alpha_i = \lambda_i(m_{ii} - 1)$ ,  $i = 1, 2, 3$ , and from formula (14) the proposition follows.  $\square$

## 2.2 3-types BPBH $Z(t) = (Z_1(t), Z_2(t), Z_3(t))$ with arbitrary lifespan d.f. $G_i(t), i = 1, 2, 3$

Recall that  $k(i) = \min(i-1, n-i)$ . In (5), (10), (13) we received  $f_i, F_i(t, s)$ ,  $m_{ii}$  for  $n$ -types BPBH. For  $n = 3$ ,  $k(1) = k(3) = 0$ ,  $k(2) = 1$ , therefore  $f_i(s_1, s_2, s_3) = 1 - \rho + \rho s_i^2; i = 1, 3$ ,  $f_2(s_1, s_2, s_3) = 1 - \rho + \rho p_{22} s_2^2 + \rho p_{23} s_1 s_3$ . We have

$$m_{12}(t) = m_{13}(t) = m_{31}(t) = m_{32}(t) = 0, \quad (21)$$

$$m_{21}(t) = 2\rho p_{22} \int_0^t m_{21}(t-u) dG_2(u) + \rho p_{23} \int_0^t m_{11}(t-u) dG_2(u), \quad (22)$$

$$m_{22}(t) = 1 - G_2(t) + 2\rho p_{22} \int_0^t m_{22}(t-u) dG_2(u), \quad (23)$$

$$m_{23}(t) = 2\rho p_{22} \int_0^t m_{23}(t-u) dG_2(u) + \rho p_{23} \int_0^t m_{33}(t-u) dG_2(u). \quad (24)$$

For the next proposition introduce the d.f.  $\widetilde{G}_i(t) = m \int_0^t e^{-\alpha_i u} dG_i(u)$ ,  $i = 1, 2, 3$ , and define the mean  $\widetilde{\mu} = \int_0^\infty u d\widetilde{G}(u)$ , assuming that the integral is finite.

**Proposition 3.** *For the asymptotic behaviour of  $\mathbf{Z}(t) = (Z_1(t), Z_2(t), Z_3(t))$  with  $\mathbf{Z}(0) = (1, 1, 1)$ , when  $t \rightarrow \infty$  we have for  $EZ_i(t), i = 1, 2, 3$ , the expressions presented in Table 1, wherein  $K_i^j, j = 1, 2, 3$ ,*



are given by

$$\begin{aligned}
K_1^i &= m_{2i}(2\rho p_{ii} - 1)/(\alpha_i 2\rho p_{ii} m_{22} \tilde{\mu}_i \tilde{\mu}_2), \\
K_2^i &= (2\rho p_{2i})/(m_{22} \tilde{\mu}_2) \int_0^\infty e^{(\delta_i - \alpha_i)u} m_{ii}(u) du, \\
K_3^i &= 2\rho p_{2i} [(2\rho p_{ii} - 1)/(\alpha_i 2\rho p_{ii} \tilde{\mu}_i)] [\widehat{G}_2(\alpha_i)/(1 - 2\rho p_{22} \widehat{G}_2(\alpha_i))], \\
\delta_i &= \alpha_i - \alpha_2, \text{ and } \widehat{G}_2(\alpha_i) = \int_0^\infty e^{-\alpha_i u} dG_2(u).
\end{aligned}$$

Table 1.

	$EZ_i(t)_{i=1,3} \sim$	$EZ_2(t) \sim$
$\delta_i = 0$	$K_1^i t e^{\alpha_2 t} + [(2\rho - 1)/(\alpha_i 2\rho \tilde{\mu}_i)] e^{\alpha_i t}$	$\frac{[(2\rho p_{22} - 1)]}{(\alpha_2 2\rho p_{22} \tilde{\mu}_2)} e^{\alpha_2 t}$
$\delta_i < 0$	$K_2^i e^{\alpha_2 t} + [(2\rho - 1)/(\alpha_i 2\rho \tilde{\mu}_i)] e^{\alpha_i t}$	$\delta_2 \equiv 0$
$\delta_i 0$	$K_3^i e^{\alpha_i t} + [(2\rho p_{ii} - 1)/(\alpha_i 2\rho \tilde{\mu}_i)] e^{\alpha_i t}$	$\delta_2 \equiv 0$

**Proof.** The asymptotic behaviour of  $Z_i(t), i = 1, 3$  depends on two Malthus parameters  $\alpha_i$  and  $\alpha_2$ . In [1] the authors studied the asymptotic behaviour of two-types BHBP with i.g.f.  $h_1(s_1, s_2) = p_0 + p_1 s_1^2 + p_2 s_2$  and  $h_2(s_2) = 1 - q + g s_2^2, 0 \leq q \leq 1$ , which appears as particular case at issue in this statement. It is not difficult to deduce for  $\{\mathbf{Z}(t) = (Z_1(t), Z_2(t), Z_3(t)) \mid (\mathbf{Z}(0) = (0, 1, 0))\}$  that  $m_{2i}(t), i = 1, 3$  satisfy

$$m_{2i}(t) \sim \begin{cases} K_1^i t e^{\alpha_2 t} & \delta_i = 0 \\ K_2^i e^{\alpha_2 t} & \delta_i < 0 \\ K_3^i e^{\alpha_i t} & \delta_i 0 \end{cases}$$

For  $i = 1, 3$   $EZ_i(t) \mid (\mathbf{Z}(0) = (1, 1, 1)) = m_{ii}(t) + m_{2i}(t)$  and  $EZ_2(t) = m_{22}(t)$ . It is well known that  $m_{ii}(t) = (m_{ii} - 1)/(\alpha_i 2\rho \tilde{\mu}_i) e^{\alpha_i t}$  for  $i = 1, 2, 3$ .  $\square$

**Corollary 1.** (Markov case) When  $G_i(t) = 1 - e^{-\lambda_i t}, i = 1, 2, 3$ , the Malthus parameters are given by  $\alpha_i = \lambda_i(m_{ii} - 1)$ . It follows from

(22)–(24) that  $m_{ii}(t) = e^{\alpha_i t}$ . Similarly for  $j = 1, 3$   $m_{2j}(t) = m_{2j} \lambda_2 t e^{\alpha_2 t}$  when  $\alpha_i = \alpha_2$  and  $m_{21}(t) = m_{21} \lambda_2 (e^{\alpha_1 t} - e^{\alpha_2 t}) / (\alpha_1 - \alpha_2)$  when  $\alpha_i \neq \alpha_2$ . Therefore, in accordance with the value of  $\delta_i = \alpha_i - \alpha_2$  we have expression for  $Em_{2i}(t), i = 1, 3$  as in Table 2.

Table 2.

	$Em_{2i}(t)_{i=1,3} \sim$	$K_j^i, j = 1, 2, 3$
$\delta_i = 0$	$K_1^i t e^{\alpha_2 t}$	$K_1^i = m_{21} \lambda_2$
$\delta_i < 0$	$K_2^i e^{\alpha_2 t}$	$K_2^i = -m_{21} \lambda_2 / \delta$
$\delta_i 0$	$K_3^i e^{\alpha_i t}$	$K_3^i = m_{21} \lambda_2 / \delta$

Furthermore we have

$$EZ_2(t) = e^{\alpha_2 t}, EZ_i(t) \mid (\mathbf{Z}(0) = (1, 1, 1)) = e^{\alpha_i t} + m_{2i}(t), i = 1, 3.$$

During the flowering of phytoplankton there is a boom in the population size. We model the phytoplankton blossoming introducing the next condition.

### Condition 2

The cell division in one of the particles gives rise to division at the same time in all other particles.

**Proposition 4.** Consider BPBH  $\{\mathbf{Z}(t) = (Z_1(t), Z_2(t), Z_3(t)) \mid \mathbf{Z}(0) = (1, 1, 1)\}$  under Condition 2. all present particles leave the same random time, i.e.  $Z(t)$  is discrete-time process  $Z_n = (Z_n^1, Z_n^2, Z_n^3), n = 1, 2, \dots$  with  $Z_0 = (1, 1, 1)$ . After a random time with probability  $\rho$  each particle generates two new, according to the probability law, given by Definition 1. Denote by  $\tau^{(k)}$  the moment of the  $k$ -th cell division,  $k = 1, 2, \dots$ . For  $E\mathbf{Z}(t) = (EZ_1(t), EZ_2(t), EZ_3(t))$  we have values as in Table 3.

Table 3.

	$E(Z_1) \equiv E(Z_3)$	$E(Z_2)$
$t \in [\tau^{(k)}, \tau^{(k+1)})$ $k=0, 1, 2, \dots$	$2^{k-1} \rho^k p_{23} (p_{22}^k - 1) / (p_{22} - 1) + (2\rho)^k$ $= 2^k \rho^k \{p_{23} (1 - p_{22}^k) / [2(1 - p_{22})] + 1\}$	$2^k \rho^k p_{22}^k$

**Proof.** Consider firstly  $\{\mathbf{Z}(t) = (Z_1(t), Z_2(t), Z_3(t)) \mid \mathbf{Z}(0) = (0, 1, 0)\}$ . The expected number of available particles  $T_1$  after each performed cell division  $EZ_1^1(t), EZ_1^2(t), \dots, EZ_1^n(t), \dots$  is:

$$E[Z_1^{(1)}(t)] = \rho p_{23},$$

$$E[Z_1^{(2)}(t)] = 2\rho^2 p_{22} p_{23} + 2\rho^2 p_{23} = 2\rho^2 p_{23}(1 + p_{22}),$$

$$E[Z_1^{(3)}(t)] = 2^2 \rho^3 p_{23}(p_{22} + 1) + 2^2 \rho^3 p_{22}^2 p_{23} = 2^2 \rho^3 p_{23}(1 + p_{22} + p_{22}^2).$$

Accept induction assumption:  $E[Z_1^{(k)}(t)] = 2^{k-1} \rho^k p_{23}(1 + p_{22} + p_{22}^2 + \dots + p_{22}^{k-1})$ . Note that  $(1 + p_{22} + p_{22}^2 + \dots + p_{22}^{k-1})$  is sum of geometric progression, then  $E[Z_1^{(k)}(t)] = 2^{k-1} \rho^k p_{23}(p_{22}^k - 1)/(p_{22} - 1)$ . Directly is obtained  $E[Z_1^{(k+1)}(t)] = 2^k \rho^{k+1} p_{23}(p_{22}^{k+1} - 1)/(p_{22} - 1)$ . Similarly we obtain  $E[Z_3^{(k)}(t)]$ . It is not difficult to deduce from here the results presented in Table 4.

Table 4.

$\{\mathbf{Z}(t) = (Z_1(t), Z_2(t), Z_3(t)) \mid (\mathbf{Z}(0) = (0, 1, 0))\}$		
	$EZ_3 \equiv EZ_1(t)$	$EZ_2(t)$
$t \in [0, \tau^{(1)})$	0	1
$t \in [\tau^{(1)}, \tau^{(2)})$	$\rho p_{23}$	$2\rho p_{22}$
$t \in [\tau^{(2)}, \tau^{(3)})$	$2\rho^2 p_{23}(1 + p_{22})$	$2^2 \rho^2 p_{22}^2$
$t \in [\tau^{(3)}, \tau^{(4)})$	$2^2 \rho^3 p_{23}(1 + p_{22} + p_{22}^2)$	$2^3 \rho^3 p_{22}^3$
...	...	...
$t \in [\tau^{(k)}, \tau^{(k+1)})$	$2^{k-1} \rho^k p_{23}[(p_{22}^k - 1)/(p_{22} - 1)]$	$2^k \rho^k p_{22}^k$
...	...	...

We have  $2^{k-1} \rho^k p_{23}(p_{22}^k - 1)/(p_{22} - 1) = 2^k \rho^k p_{23}(1 - p_{22}^k)/[2(1 - p_{22})]$  and from Table 4., if  $\mathbf{Z}(0) = (1, 1, 1)$ , we receive Table 3.  $\square$

**Proposition 5.** Consider  $\mathbf{Z}(t) = (Z_1(t), Z_2(t), Z_3(t)); \mathbf{Z}(0) = (1, 1, 1)$  under Condition 2. Let  $p_{23} = 2p_{22}(= \frac{2}{3})$  and  $\rho = \frac{1}{2}$ . We can reduce  $\mathbf{Z}(t)$  to three 1 - type processes denoted by  $V_i(t)$ ,  $i = 1, 2, 3$ . Every particle of  $V_i(t)$  has a random lifespan and at his end either produces a new particle of the same type [11], with probability  $p_{22}$  or dies without leaving a generation.  $\tau^{(k)}$  is the moment of  $k$ -th division in

each of the processes. Additionally a newborn particle immigrated at any moment  $\tau^{(k)}$ ,  $k = 1, 2, \dots$  to the process  $V_i(t)$  for  $i = 1, 3$ . For  $\{E(\mathbf{Z}(t) \mid \mathbf{Z}(0) = (1, 1, 1) = (E(V_1(t) \mid V_1(0) = 1), \dots, E(V_3(t) \mid V_3(0) = 1), k = 0, 1, \dots, \text{the results presented in Table 5. are correct.}$

Table 5.

	$EZ_1(t) \equiv EZ_3(t)$	$EZ_2(t)$
$t \in [\tau^{(k-1)}, \tau^{(k)})$	$\frac{1-p_{22}^{k+1}}{1-p_{22}} = 1 + p_{22} + \dots + p_{22}^k$	$p_{22}^k$

**Proof.** Directly from Proposition 4, when  $\rho = 1/2$ , we obtain Table 6.

Table 6.

	$EZ_1(t) \equiv EZ_3(t)$	$EZ_2(t)$
$t \in [0, \tau^{(1)})$	$p_{22} + 1$	$p_{22}$
$t \in [\tau^{(1)}, \tau^{(2)})$	$p_{22}(1 + p_{22}) + 1$	$p_{22}^2$
$t \in [\tau^{(2)}, \tau^{(3)})$	$p_{22}(1 + p_{22} + p_{22}^2) + 1$	$p_{22}^3$
$\dots$	$\dots$	$\dots$
$t \in [\tau^{(k-1)}, \tau^{(k)})$	$\frac{1-p_{22}^{k+1}}{1-p_{22}} = 1 + p_{22} + \dots + p_{22}^k$	$p_{22}^k$
$\dots$	$\dots$	$\dots$

### 3 Model simulation of the diatom population

In this section we simulate model  $\mathbf{Z}(t) = (Z_1(t), Z_2(t), Z_3(t)) \mid (\mathbf{Z}(0) = (1, 1, 1))$  under Condition 2. with normal distributed lifespan and  $\rho = 1/4, 1/2, 5/9$ . For the simulations we propose an algorithm, realized using MATLAB. Recall that the birth sizes are  $r(T) \in [d^i; d^{\text{“}}]$  and we divided this interval in  $n$  equal subintervals. According to Definition 1 if  $n = 3$  practically we could accept  $[d^i; d^{\text{“}}] = 6\sigma$  because of the  $3\sigma$  rule of the  $N(\mu, \sigma)$ . For the diatoms, one of the most common species phytoplankton  $d^i = 30\mu\text{m}$ ,  $d^{\text{“}} = 150\mu\text{m}$ . We get

$\sigma = 120/6 = 20$ . Then  $p_{21} = p_{23} \approx 0,1587$  and  $p_{22} \approx P(\mu - \sigma < r(T^*_{*2}) < \mu + \sigma) = 2\Phi(1) = 0,6826$ , where  $\Phi(x)$  is the Laplace function. From Proposition 4. we could get the results, presented in Table 7.

Table 7.

$\mathbf{Z}(t) = (Z_1(t), Z_2(t), Z_3(t)) \mid (\mathbf{Z}(0) = (1, 1, 1))$		
$k = 0, 1, 2, \dots$	$EZ_1(t) \equiv EZ_3(t)$	$EZ_2(t)$
$t \in [\tau^{(k)}, \tau^{(k+1)})$	$\frac{0,1587(2\rho)^k(1 - 0,6826^k)}{0,6348 + (2\rho)^k}$	$0,6826^k(2\rho)^k$

We offer algorithm like the given in [9],  $N$  - natural

(Step 1.)  $k := 1, m := 0$ .

(Step 2.) Generate  $2(0,1587(2\rho)^k(1 - 0,6826^k)/0,6348 + (2\rho)^k) + 0,6826^k(2\rho)^k$  normal distributed random variables (lifespans).

(Step 3.) Find the minimum of the generated numbers in Step 2.

(Step 4.) Determine the endpoints of the intervals  $[\tau^{(k)}, \tau^{(k+1)})$ .

(Step 5.)  $m := m + 1$ .

$\langle m \leq N \rangle$  *Yes*  $k := k + 1$  go to Step 2  
*No* go to Step 6

(Step 6.) Plot result.

(Step 7.) End.

Figure 2, Figure 3 and Figure 4 represent the EZ for number of cell divisions  $k = 1 \dots, N$  over the axis with three values of  $\rho$ :  $\rho = 1/4$ ;  $\rho = 1/2$  and  $\rho = 5/9$ .

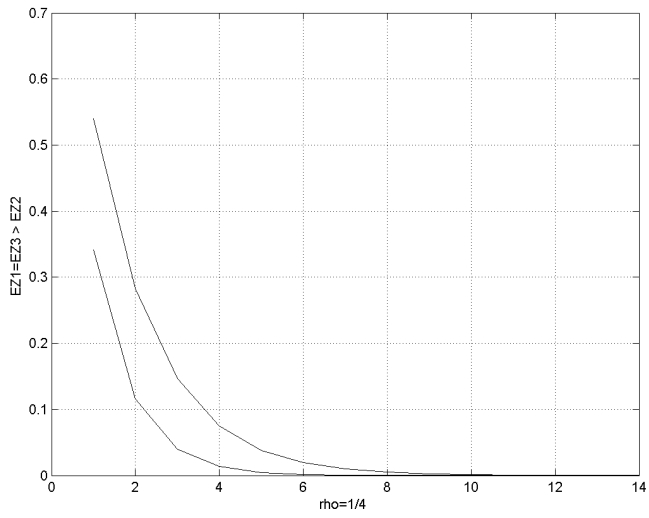


Figure 2: EZ for  $\rho = 1/4$

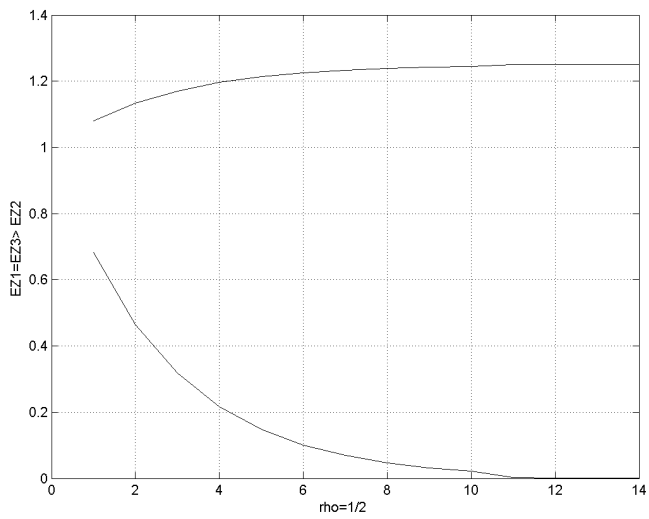


Figure 3: EZ for  $\rho = 1/2$

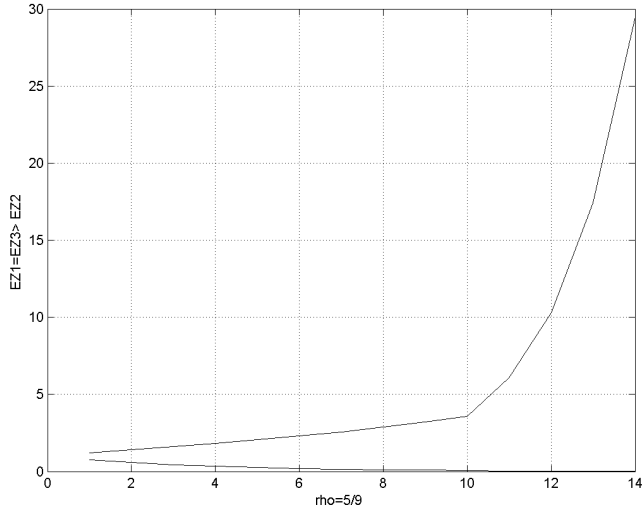


Figure 4: EZ for  $\rho = 5/9$

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### Conclusion

Phytoplankton is one of the most ancient inhabitants of our planet, playing key role at the ocean and marine food chains.

It is modelled the dynamics of phytoplankton population, usually modeled by differential equations. It is proposed, as a modelling tool, multitype branching stochastic process of Bellman-Harris.

The proposed model could be applicable for a random populations of unicellular organisms, but also for arbitrary sets of objects which operate a certain period of time and then split into two newborn objects. Data on the amount of chlorophyll-a may be obtained by satellite and the chlorophyll-a quantity in a unit biomass phytoplankton is known. It is possible to consider as a single particle in the proposed model, not the whole cell phytoplankton, only the chlorophyll-a quantity, say chlorophyll-unit. We can draw some conclusions about the phyto-

plankton amount dynamics, considering specific to any particular area number of phytoplankton species and corresponding correlation coefficients between phytoplankton and chlorophyll-a. On this basis the obtained processes could help us to make accurate prediction about the expected number of particles and its asymptotic behaviour in the infinity.

Given any mechanisms to influence phytoplankton's population size we could possibly increase fish production for example. Phytoplankton is of importance in ecological sense, climate regulation and feeding people.

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