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Some Generalizations of Logistic Model

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Abstract. During the last decades models have been created to achieve the description and the representation of phenomena and biological populations. These models are often an ordinary differential equations. A new population models are proposed in this article. This models are named the Generalized Logistic Models and it is an improvement-generalization of the already known, from the bibliography, Logistic Equation (Model). The solution of the new model is presented as well as its graph drown with Mathematica 11.0 and properties for a more complete and spherical view.

1 Introduction

A population is a group of organisms of the same species (fishes, birds etc.) that live in a particular area. The number of organisms in a population changes over time because of births, deaths, emigration, immigration and some outside factors. Of course, births and immigration increase the size of the population, on the other hand deaths

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and emigration for example decrease its size. The increase in the number of organisms in a population is mentioned as population growth. There are factors that can help populations grow and others than can slow down populations from growing. Factors that limit population growth are called limiting factors.

A model is simply a system of organisms, information or things presented as a mathematical description of an entity. Thus Population Models are approximations of reality described by mathematical formulae (differential equations for example) or computer programs and packages. Population Models are usually created and developed to predict the behaviour of ecological systems and biological populations. As it can be observed Population Models can be very useful and interesting, because they represent reality and some specific data and because they are capable, in many occasions, to give accurate and precise estimates that can help human mind to predict the future of a population and to compare the results gained with similar results from other models or from different populations [1].

2 The Logistic Equation

The Logistic Equation is widely used and very often seen, because it has lots of applications and can be very useful in various models. Some characteristic applications of the logistic equation is the population growth problem (model) and the harvesting in a biological population problem (model) [2].

Let us suppose that $x(t)$ denotes the size of a biological population at time t and $x'(t)$ denotes the increase in the population. The carrying capacity K of the biological environment is the maximum population that the environment can stand. If the population equals this carrying capacity then the deaths among the population become more than the births and the population can not grow any more. Let us suppose that a is the (positive) birth rate, which is not dependent of N and b is the (positive) crowding coefficient which depends on the carrying capacity of the population [3]. So it is true that :

$$x'(t) = a\left(1 - \frac{x(t)}{K}\right)x(t) \qquad (2.1)$$

where $\frac{a}{K} = b$.

2.1 Properties of the logistic equation

Equation (2.1) has equilibrium points when $x'(t) = 0$, that is when $x = 0$ and $x = K$. Thus we can say that:

$0 < x < K$ gives $x' > 0$ and also that $x > K$ gives $x' < 0$.

So it can be observed that the point $x = 0$ is unstable since any positive initial population will increase monotonically and try to reach $x = K$. On the other hand, the point $x = K$ is asymptotically stable since if there is a small displacement from the point, the population will tend again towards it [4].

2.2 The analysis and the graph of the function $F(x) = rx\left(1 - \frac{x}{K}\right)$

Let $F(x) = rx\left(1 - \frac{x}{K}\right)$. It is $F(x) = rx\left(1 - \frac{x}{K}\right) = -\frac{r}{K}x^2 + rx$, which is a parabola with respect to x. The maximum value is for $x = \frac{K}{2}$, and there are two roots $x_1 = 0$, $x_2 = K$. The position of the maximum is symbolized by $xMSY$, while the maximum value of the function $(= 0.25Kr)$ is symbolized by MSY.

The graph of $F(x)$ in the interval $[0, K]$ is depicted in Figure 1.

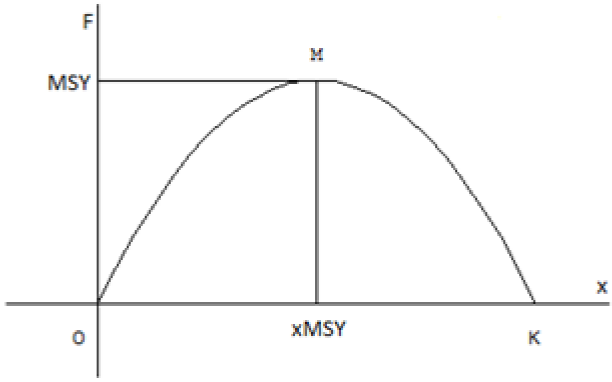

Figure 1. Graph of the function $F(x)$

The roots of $F(x)$ are 0 and K, therefore the functions $x(t) = 0$ and $x(t) = K$ are the equilibrium solutions of the differential equation.

The graph of $F(x)$ with $r = 0.2$ and $K = 1, 2, 3, \ldots, 12$ is depicted in Figure 2.

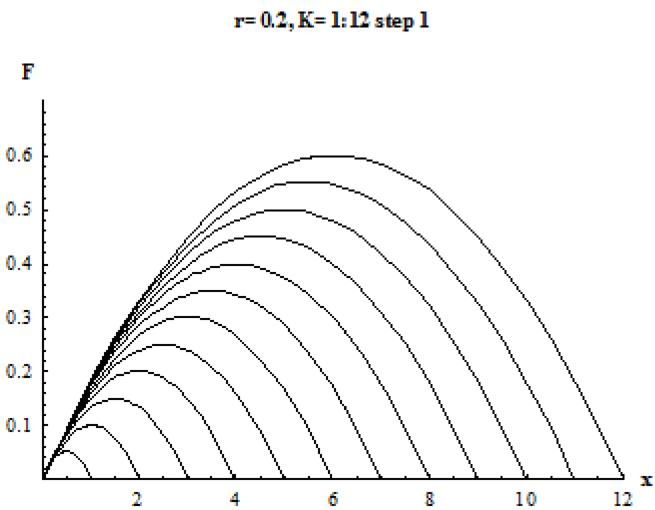

Figure 2. Graph of the function $F(x)$ with $r = 0.2$ and $K = 1, 2, 3, \ldots, 12$

Similarly the graph of $F(x)$ with $K = 12$ and $r = 0.1, 0.2, 0.3, \dots, 1$ is depicted in Figure 3.

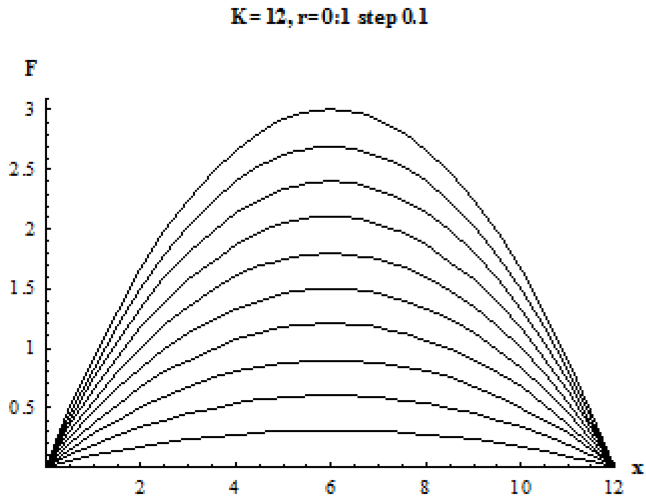

Figure 3. Graph of the function $F(x)$ with $K = 12$ and $r = 0.1, 0.2, 0.3, \dots, 1$

2.3 The solution of the differential equation $\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right)$ with $x(0) = x_0$

It is $\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right)$ with $x(0) = x_0$, therefore

$$\frac{dx}{dt} = \frac{-r}{K} x(x - K) \Rightarrow -\frac{1}{K}\left(\frac{1}{x} - \frac{1}{x - K}\right) dx = -\frac{r}{K} dt$$

$$\Rightarrow \int \left(\frac{1}{x} - \frac{1}{x - K}\right) dx = \int r dt$$

$$\Rightarrow \ln(x) - \ln(x - K) = rt + c_1$$

$$\Rightarrow \frac{x}{x - K} = e^{rt + c_1} \quad or \quad x(t) = K \frac{e^{rt} e^{c_1}}{e^{rt} e^{c_1} - 1}.$$

For $t = 0$, we get $x(0) = K\frac{e^{c_1}}{e^{c_1}-1} \Rightarrow e^{c_1} = \frac{x_0}{x_0-K}$, thus $x(t) = K\frac{e^{rt}}{\frac{K-x_0}{x_0}+e^{rt}}$.

Let $c = \frac{K-x_0}{x_0}$ then $x(t) = K\frac{e^{rt}}{c+e^{rt}}$, therefore

$$x(t) = K\frac{1}{1+ce^{-rt}},\ t \geq 0. \tag{2.2}$$

The function $x(t)$ which is defined by equation (2.2) is the solution of the differential equation.

2.4 Analysis of the function $x(t) = K\frac{1}{1+ce^{-rt}}$

It is $x(t) = K\frac{x_0e^{rt}}{K-x_0+x_0e^{rt}}$

- If $x_0 < K \Rightarrow K - x_0 > 0 \Rightarrow x(t) < K, \forall t > 0$
- If $x_0 > K \Rightarrow K - x_0 < 0 \Rightarrow x(t) > K, \forall t > 0$
- If $x_0 = K \Rightarrow x(t) = K, \forall t > 0$

and $\lim_{t\to+\infty} x(t) = K$.

Concluding $x(t) = K\frac{1}{1+ce^{-rt}}$ and thus $\lim_{t\to+\infty} x(t) = K$, therefore K is a stable equilibrium. On the other hand 0 is an unstable equilibrium [5].

The graph of $x(t)$ for various initial values is depicted in Figure 4 and Figure 5.

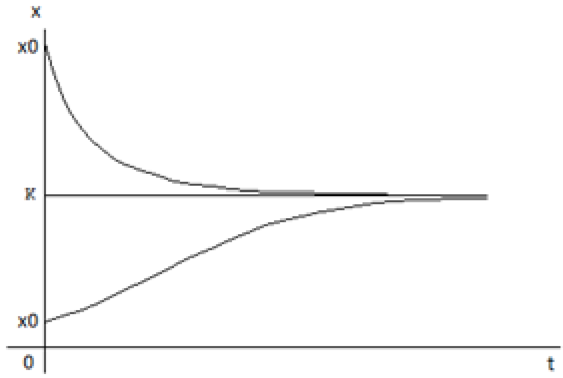

Figure 4. Graph of the solutions $x(t)$

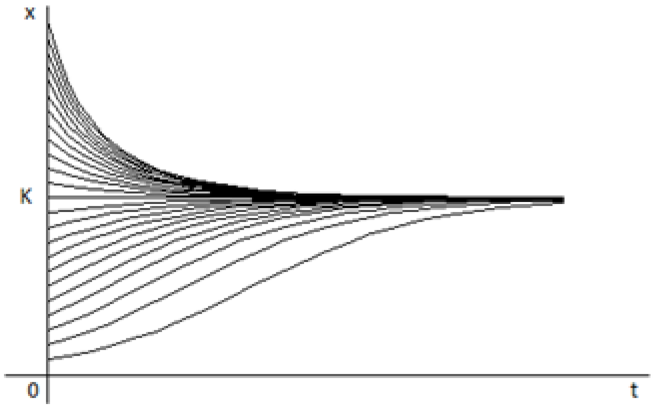

Figure 5. Graph of the solutions $x(t)$

3 First Generalization of Logistic Model

The function $F(x) = rx\left[1 - \left(\frac{x}{K}\right)^b\right]$ is introduced, with $b > 0$.

Therefore the (logistic) differential equation takes the form

$$\frac{dx}{dt} = rx\left[1 - \left(\frac{x}{K}\right)^b\right] = F(x) \tag{3.1}$$

which is named the Generalized Logistic Model.

3.1 The analysis and the graph of the function $F(x) = rx\left[1 - \left(\frac{x}{K}\right)^b\right]$

Let $F(x) = rx\left[1 - \left(\frac{x}{K}\right)^b\right] = -\frac{r}{K^b}x^{1+b} + rx$, therefore it is true that $F'(x) = -\frac{r(1+b)}{K^b}x^b + r$. The root of $F'(x)$ is $x = \frac{K}{(1+b)^{\frac{1}{b}}}$.

The graph of $F(x)$ in the interval $[0, K]$ is depicted in Figure 6.

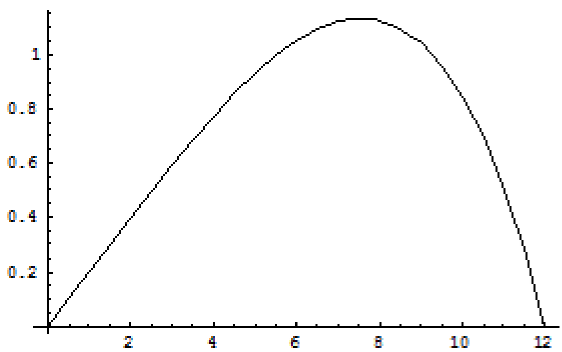

Figure 6. Graph of the function $F(x)$ with $b = 3$, $r = 0.2$ and $K = 12$

The roots of $F(x)$ are 0 and K, therefore the functions $x(t) = 0$ and $x(t) = K$ are the equilibrium solutions of the differential equation.

The graph of $F(x)$ with $r = 0.2$, $K = 12$, and $b = 0.5, \ldots, 5.5$ step 0.5 is depicted in Figure 7.

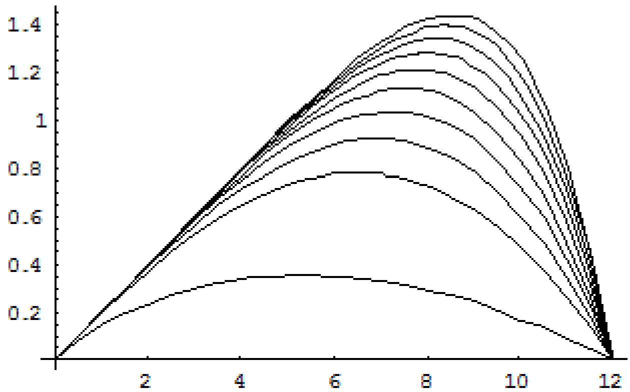

Figure 7. Graph of the function $F(x)$ with $r = 0.2$, $K = 12$ and $b = 0.5, \ldots, 5.5$ step 0.5

3.2 The solution of the differential equation $\frac{dx}{dt} = rx\left[1 - \left(\frac{x}{K}\right)^b\right]$ with $x(0) = x_0$

It is $\frac{dx}{dt} = rx\left[1 - \left(\frac{x}{K}\right)^b\right]$ with $x(0) = x_0$, therefore

$$\frac{dx}{dt} = \frac{-r}{K^b} x(x^b - K^b)$$

$$\Rightarrow \frac{dx}{x(x^b - K^b)} = \frac{-r}{K^b} dt \Rightarrow -\frac{1}{K^b}\left[\int \frac{dx}{x} - \frac{1}{b}\int \frac{dx^b}{x^b - K^b}\right] = -\int \frac{r}{K^b} dt$$

$$\Rightarrow \ln x^b - \ln(x^b - K^b) = brt + c_1 \Rightarrow \frac{x^b}{x^b - K^b} = c_2 e^{brt} \quad or$$

$$x = K e^{rt} \sqrt[b]{\frac{c_2}{c_2 e^{brt} - 1}}.$$

For $t = 0$ we get $x(0) = K\sqrt[b]{\frac{c_2}{c_2-1}} = x_0 \Rightarrow c_2 = \frac{x_0^b}{x_0^b - K^b}$, thus

$$x(t) = ke^{rt}\sqrt[b]{\frac{\frac{x_0^b}{x_0^b-K^b}}{e^{brt}\frac{x_0^b}{x_0^b-K^b} - 1}} = \frac{Ke^{rt}}{\sqrt[b]{\frac{K^b-x_0^b}{x_0^b} + e^{brt}}}.$$

Let $c = \frac{K^b - x_0^b}{x_0^b}$ then $x(t) = K\frac{e^{rt}}{\sqrt[b]{c+e^{rt}}}$, therefore

$$x(t) = K\frac{1}{\sqrt[b]{1 + ce^{-brt}}}, \quad t \geq 0 \tag{3.2}$$

The function $x(t)$ which is defined by equation (3.2) is the solution of the differential equation.

3.3 Analysis of the function $x(t) = K\frac{1}{\sqrt[b]{1+ce^{-brt}}}$

It is $x(t) = K\frac{e^{rt}x_0}{\sqrt[b]{K^b - x_0^b + x_0^b e^{brt}}}$.

- If $x_0 < K \Rightarrow K^b - x_0^b > 0 \Rightarrow x(t) < K, \forall t > 0$
- If $x_0 > K \Rightarrow K^b - x_0^b < 0 \Rightarrow x(t) > K, \forall t > 0$
- If $x_0 = K \Rightarrow x(t) = K, \forall t > 0$

and $\lim_{t \to +\infty} x(t) = K$.

Concluding $x(t) = K\frac{1}{\sqrt[b]{1+ce^{-brt}}}$ and thus $\lim_{t \to +\infty} x(t) = K$, therefore K is a stable equilibrium. On the other hand is an unstable equilibrium.

The graph of $x(t)$ for various initial values is depicted in Figure 8.

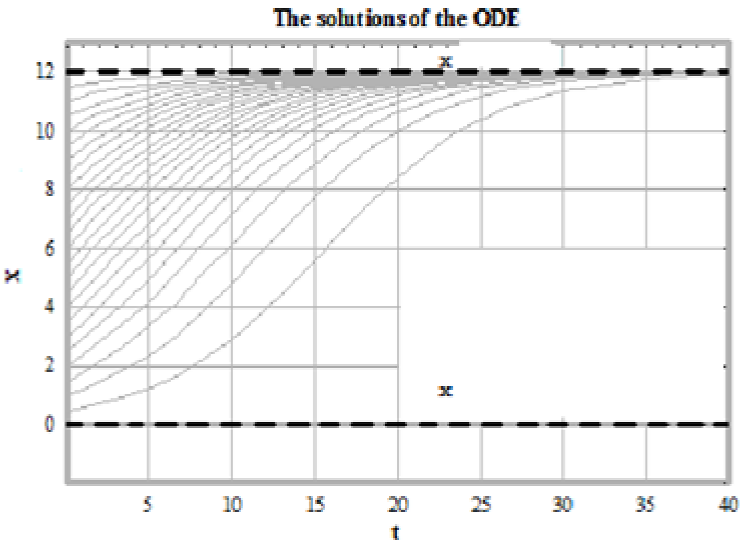

Figure 8. Graph of the solutions $x(t)$ of the Generalized Logistic Equation

4 Second Generalization of Logistic Model

Consider now the function $F(x) = kx^b(a - x) - \chi x^{b+1}$ with $b \geq 1$, $a > 0$ and $\chi > 0$. The equation $\frac{dx}{dt} = F(x)$ also seems to be of interest for biomathematics.

Therefore the (logistic) differential equation

$$\frac{dx}{dt} = rx^b\left(1 - \frac{x}{K}\right) = F(x) \tag{4.1}$$

where $r = ka$ and $K = \frac{ka}{k+\chi}$ is another generalization of Logistic Model.

4.1 The analysis and the graph of the function $F(x) = rx^b\left(1 - \frac{x}{K}\right)$.

Let $F(x) = rx^b\left(1 - \frac{x}{K}\right) = -\frac{r}{K}x^{1+b} + rx^b$, therefore it is true that $F'(x) = rbx^{b-1} - \frac{r(1+b)}{K}x^b$. The roots of $F'(x)$ are $x = 0$ and $x = \frac{b}{1+b}K$.

The graph of $F(x)$ in the interval $[0, K]$ is depicted in Figure 9.

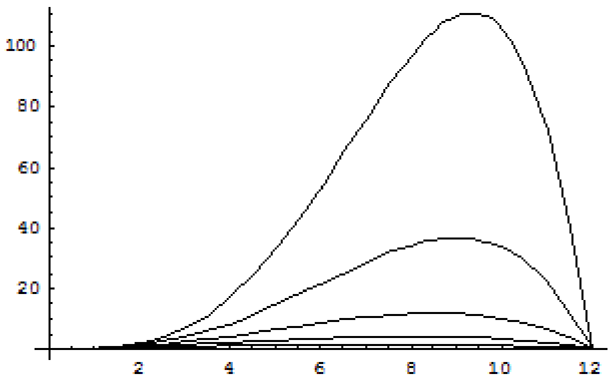

Figure 9. Graph of the function $F(x)$ with $r = 0.2$, $K = 12$ and $b = 1.5, \ldots, 4.0$ step 0.5

4.2 The solution of $\frac{dx}{dt} = rx^b\left(1 - \frac{x}{K}\right)$ with $x(0) = x_0$ and natural b

It is $\frac{dx}{dt} = rx^b\left(1 - \frac{x}{K}\right)$ with $x(0) = x_0$, therefore

$$\frac{dx}{dt} = -r\frac{r}{K}x^b(x - K) \Rightarrow \frac{dx}{x^b(x - K) = -\frac{r}{K}dt}$$

Denote $\mathfrak{I}_b(x) = \int \frac{dx}{x^b(x-K)}$

$$\Rightarrow \mathfrak{I}_b(x) = -\frac{1}{K}\left[\int \frac{dx}{x^b} - \int \frac{dx}{x^{b-1}(x - K)}\right] = -\frac{1}{K}\left[\int \frac{dx}{x^b} - \mathfrak{I}_{b-1}(x)\right].$$

Recursively

$$\mathfrak{I}_b(x) = -\frac{1}{K}\left[\int \frac{dx}{x^b} - \mathfrak{I}_{b-1}(x)\right] =$$

$$-\frac{1}{K}\left[\int \frac{dx}{x^b} - \frac{1}{K}\left[\int \frac{dx}{x^{b-1}} - \cdots + \frac{1}{K}\mathfrak{I}_1(x)\right] \ldots\right],$$

where $\mathfrak{I}_1(x) = \int \frac{dx}{x} - \int \frac{dx}{x-K} = \ln \frac{x}{x-K}$.

So for $b = 2$ we have $\mathfrak{I}_2(x) = \frac{1}{Kx} - \frac{1}{K^2} \ln \frac{x}{x-K}$,

for $b = 3$ we have $\mathfrak{I}_3(x) = \frac{1}{2Kx^2} + \frac{1}{K^2 x} - \frac{1}{K^3} \ln \frac{x}{x-K}$,

for $b = 4$ we have $\mathfrak{I}_4(x) = \frac{1}{3Kx^3} + \frac{1}{2K^2x^2} + \frac{1}{K^3 x} - \frac{1}{K^4} \ln \frac{x}{x-K}$, etc.

Suppose that $b = n$, then

$$\mathfrak{I}_n(x) = \sum_{j=1}^{n-1} \frac{1}{jK^{n-j}x^j} - \frac{1}{K^n} \ln \frac{x}{x-K} = \frac{1}{K^n}\left[\sum_{j=1}^{n-1} \frac{1}{j}\left(\frac{K}{x}^j - \ln \frac{1}{1-\frac{K}{x}}\right)\right].$$

Thus the solution of (4.1) with $x(0) = x_0$ can be represented in the form

$$t = \frac{1}{rK^{n-1}}\left[\ln \frac{1-\frac{K}{x_0}}{1-\frac{K}{x}} - \sum_{j=1}^{n-1} \frac{1}{j}\left(\left(\frac{K}{x}\right)^j - \left(\frac{K}{x_0}\right)^j\right)\right] \qquad (4.2)$$

For $b = n = 2$ the dependence of t and x, i.e. $rt = \frac{1}{K} \ln \frac{1-\frac{K}{x_0}}{1-\frac{K}{x}} - \frac{1}{x} + \frac{1}{x_0}$ is depicted in the Figures 10, 11.

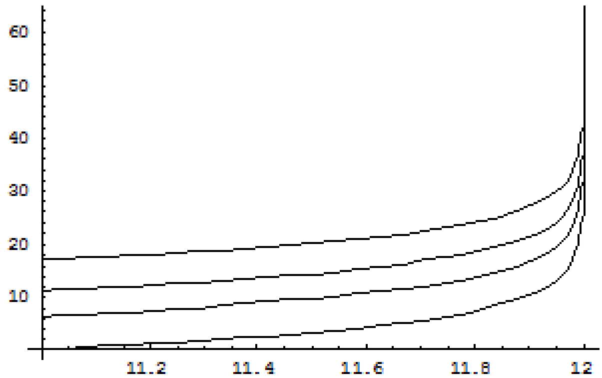

Figure 10. Graph for $r = 0.02$, $K = 12$ and $x_0 = 5, 7, 9, 11$ $(x_0 < K)$

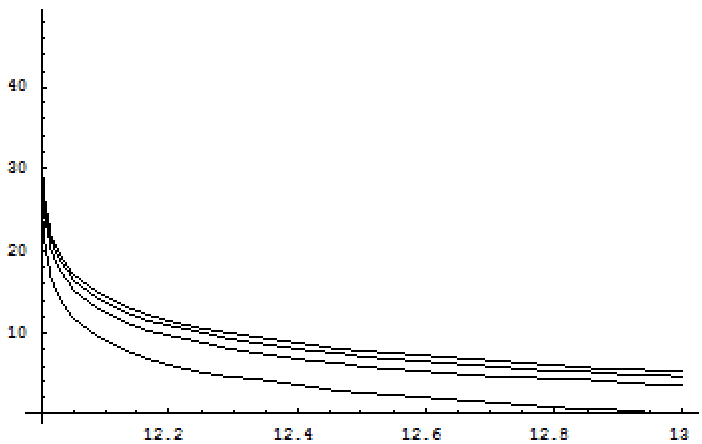

Figure 11. Graph for $r = 0.02$, $K = 12$ and $x_0 = 19, 17, 15, 13$ $(x_0 > K)$

Remember that $K = \frac{Ka}{K+x}$, so for any natural b

- If $x_0 < K \Rightarrow 1 - \frac{K}{x_0} < 0 \Rightarrow 1 - \frac{K}{x} < 0 \Rightarrow x(t) < \frac{ka}{k+\chi}, \forall t > 0$
- If $x_0 > K \Rightarrow 1 - \frac{K}{x_0} < 0 \Rightarrow x(t) > \frac{ka}{k+\chi}, \forall t > 0$
- If $x_0 = K \Rightarrow x(t) = \frac{ka}{k+\chi}, \forall t > 0$

and $\lim_{t \to +\infty} x(t) = \frac{ka}{k+\chi}$.

Therefore K is a stable equilibrium. On the other hand 0 is an unstable equilibrium.

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