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A new transmuted cumulative distribution function based on the Verhulst logistic function with application in population dynamics

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Abstract In this note we find application of a new class cumulative distribution function transformations to construct a family of sigmoidal functions based on the Verhulst logistic function.

We prove estimates for the Hausdorff approximation of the shifted Heaviside step function by means of this family. Numerical examples, illustrating our results are given.

Keywords Cumulative distribution function, Logistic function, Shifted Heaviside step function, Hausdorff distance, Upper and lower bounds.

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1 Introduction

In literature, several transformations exists to obtain a new cumulative distribution function (cdf) using other(s) well-known cdf(s) [1]–[7].

Many researches have used the quadratic rank transmuted map (QRTM) to develop new life time distribution.

Kumar et al. [8] proposed the cdf distribution by the use of any two cdf $F_1(t)$ and $F_2(t)$ of baseline continuous distribution(s) with common spectrum, by the transformation:

Definition 1. *[8]*

$$G(t) = \frac{F_1(t) + F_2(t)}{1 + F_1(t)}. \tag{1}$$

If $F_1(t) = F_2(t) = F(t)$, then (1) reduces to the following form

$$G(t) = \frac{2F(t)}{1 + F(t)}. \tag{2}$$

The transformation (1) has great applications in life time analysis.

Definition 2. *Define the logistic (Verhulst) function f on $\mathbb{R}$ as*

$$f(t) = \frac{1}{1 + e^{-kt}}. \tag{3}$$

The logistic function belongs to the important class of smooth sigmoidal functions arising from population and cell growth models.

Since then the logistic function finds applications in many scientific fields, including biology, population dynamics, chemistry, demography, economics, geoscience, mathematical psychology, probability, financial mathematics, statistics, insurance mathematics, nucleation theory to name a few [9]–[18].

Definition 3. *The (interval) step function is:*

$$h_{t_0}(t) = \begin{cases} 0, & \text{if} \quad t < t_0, \\ [0,1], & \text{if} \quad t = t_0, \\ 1, & \text{if} \quad t > t_0, \end{cases}$$

usually known as shifted Heaviside step function.

Definition 4. *[19], [20] The Hausdorff distance (the H–distance) $\rho(f,g)$ between two interval functions f, g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,*

$$\rho(f,g) = \max\{ \sup_{A \in F(f)} \inf_{B \in F(g)} ||A - B||, \sup_{B \in F(g)} \inf_{A \in F(f)} ||A - B||\}, \quad (4)$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t,x)|| = \max\{|t|, |x|\}$; hence the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in $\mathbb{R}^2$ is $||A - B|| = max(|t_A - t_B|, |x_A - x_B|)$.

In this paper we discuss several computational, modelling and approximation issues related to the class of cdf transformation (2) to construct a family of sigmoidal functions based on the Verhulst logistic function.

2 Main Results

Let us consider the following sigmoid

$$G(t) = \frac{2\frac{1}{1+e^{-kt}}}{1 + \frac{1}{1+e^{-kt}}} \quad (5)$$

with

$$G(t_0) = \frac{1}{2}, \quad t_0 = -\frac{1}{k}\ln 2 \quad (6)$$

based on (2) with the Verhulst logistic function $f(t)$.

The H-distance $d = \rho(h_{t_0}, G)$ between the shifted Heaviside step function h_{t_0} and the sigmoidal function G satisfies the relation:

$$G(t_0 + d) = \frac{2\frac{1}{1+e^{-k(t_0+d)}}}{1 + \frac{1}{1+e^{-k(t_0+d)}}} = 1 - d. \tag{7}$$

The following theorem gives upper and lower bounds for $d = d(k)$

Theorem 2.1 *The H-distance $d(k)$ between the function h_{t_0} and the function G can be expressed in terms of the rate parameter k for any real $k \geq 2$ as follows:*

$$d_l = \frac{1}{2.5(1 + 0.25k)} < d < \frac{\ln 2.5(1 + 0.25k)}{2.5(1 + 0.25k)} = d_r. \tag{8}$$

Proof. We define the functions

$$F_1(d) = \frac{2\frac{1}{1+e^{-k(t_0+d)}}}{1 + \frac{1}{1+e^{-k(t_0+d)}}} - 1 + d \tag{9}$$

$$G_1(d) = -\frac{1}{2} + (1 + 0.25k)d. \tag{10}$$

From Taylor expansion

$$\frac{2\frac{1}{1+e^{-k(t_0+d)}}}{1 + \frac{1}{1+e^{-k(t_0+d)}}} - 1 + d - (-\frac{1}{2} + (1 + 0.25k)d) = O(d^2)$$

we see that the function $G_1(d)$ approximates $F_1(d)$ with $d \to 0$ as $O(d^2)$ (cf. Fig.1).

In addition $G_1'(d) > 0$ and for $k \geq 2$

$$G_1(d_l) < 0; \quad G_1(d_r) > 0.$$

This completes the proof of the inequalities (8).

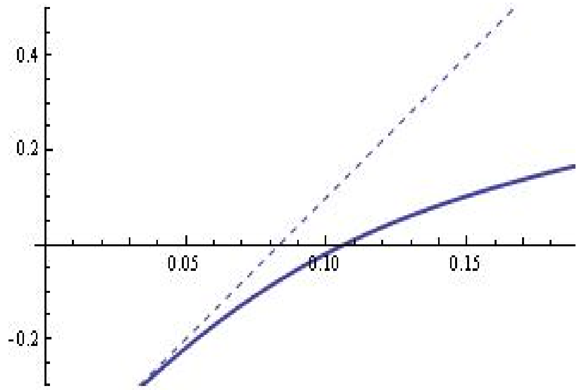

Figure 1: The functions $F_1(d)$ and $G_1(d)$ for $k = 20$.

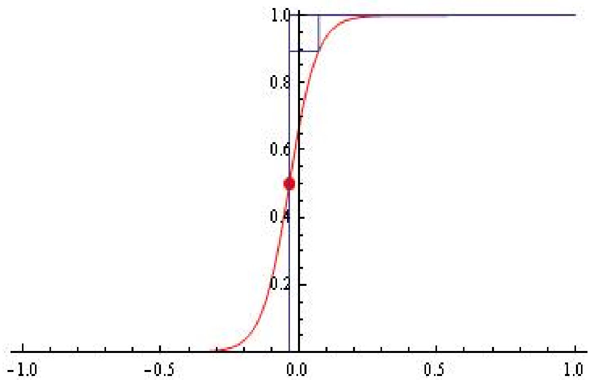

Figure 2: The H-distance $d(k)$ between the functions h_{t_0} and G for $k = 20$ is $d = 0.106402$; $d_l = 0.066667$; $d_r = 0.180537$; $t_0 = -0.0346574$.

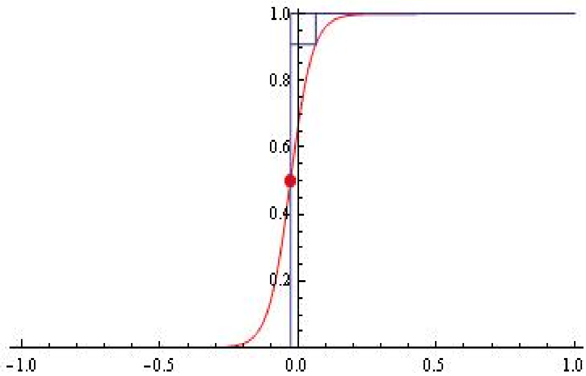

Figure 3: The H-distance $d(k)$ between the functions h_{t_0} and G for $k = 25$ is $d = 0.0917149$; $d_l = 0.0551724$; $d_r = 0.159851$; $t_0 = -0.0277259$.

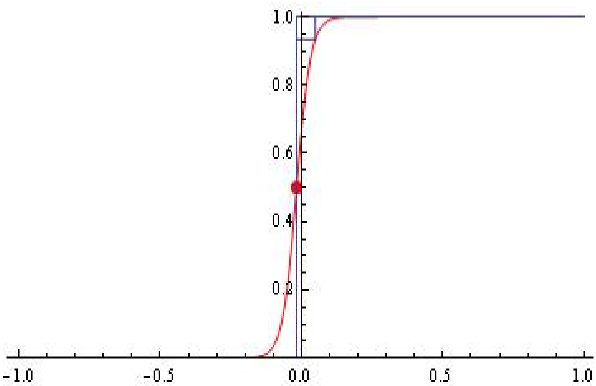

Figure 4: The H-distance $d(k)$ between the functions h_{t_0} and G for $k = 40$ is $d = 0.0661748$; $d_l = 0.0363636$; $d_r = 0.120516$; $t_0 = -0.0173287$.

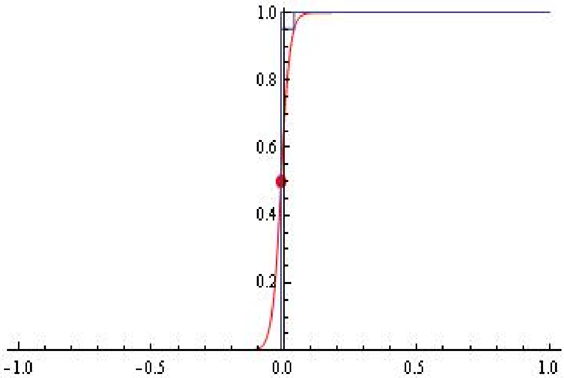

Figure 5: The H-distance $d(k)$ between the functions h_{t_0} and G for $k = 60$ is $d = 0.0493157$; $d_l = 0.025$; $d_r = 0.092222$; $t_0 = -0.0115525$.

The generated sigmoidal functions $G(t)$ for $k = 20, 25, 40, 60$ are visualized on Fig. 2–Fig.5

Some computational examples using relations (8) are presented in Table 1. The third column of Table 1 contains the value of d for prescribed values of k computed by solving the nonlinear equation (7).

3 Conclusions

To achieve our goal, we obtain new estimates for the H-distance between a shifted Heaviside step function and its best approximating family of transmuted cumulative distribution function $G(t)$ based on the Verhulst logistic function.

The result has application in population dynamics.

k	d_l	d computed *by* (7)	d_r
10	0.114286	0.163351	0.247892
20	0.066667	0.106402	0.180537
30	0.0470588	0.0809732	0.143829
40	0.0363636	0.0661748	0.120516
60	0.025	0.0493157	0.092222
100	0.0153846	0.0335928	0.0642213
500	0.0031746	0.00933024	0.0182621
1000	0.00159363	0.00524519	0.0102657

Table 1: Bounds for $d(k)$ computed by (7) and (8) for various rates k

Numerical examples, illustrating our results are given.

For other results, see [21]–[29].

We propose a software module within the programming environment *CAS Mathematica* for the analysis of the considered families of transmuted cumulative distribution functions.

The module offers the following possibilities:

- generation of the function $G(t)$ under user defined values of the reaction rate k and t_0;

- calculation of the H-distance between the Heaviside function h_{t_0} and the sigmoidal function $G(t)$;

- software tools for animation and visualization.

Remarks. In [30] we consider popular transformations [5], [7]:

$$G1(t) = \frac{1}{e-1}\left(e^{F(t)} - 1\right) \tag{11}$$

$$G2(t) = e^{1-\frac{1}{F(t)}}. \tag{12}$$

for generating of some sigmoidal functions based on the Verhulst logistic function.

```
Manipulate[Dynamic@Show[Plot[G[t], {t, -3, 3}, LabelStyle → Directive[Green, Bold],
    PlotLabel → 2 * (1 / (1 + Exp[-k * t])) / (1 + 1 / (1 + Exp[-k * t]))],
   PlotRange → {Automatic, {0, 1}}, AxesOrigin → {0, 0}],
 {{k, 1}, 1, 100, Appearance → "Open"},
 {{t0, 0}, -0.001, 10, Appearance → "Open"},
 Initialization :→ (G[t_] := 2 * (1 / (1 + Exp[-k * t])) / (1 + 1 / (1 + Exp[-k * t])))]
```

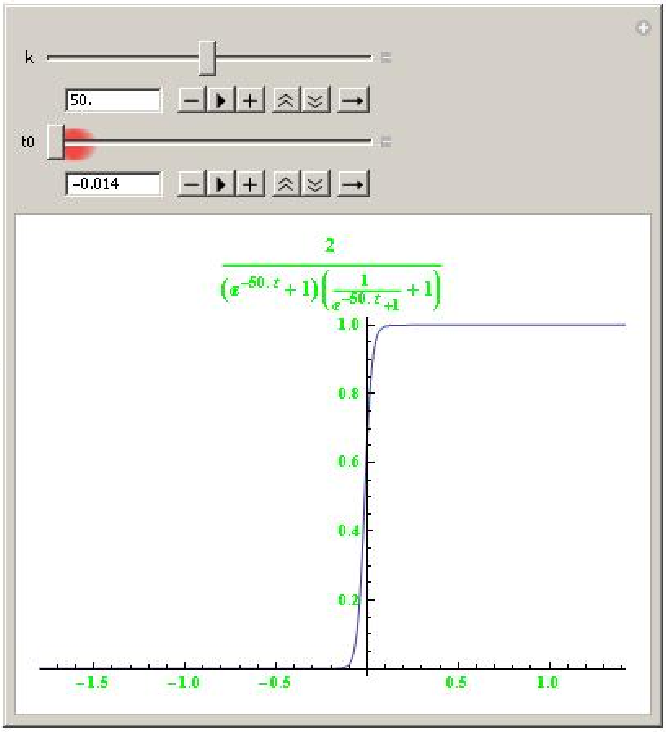

Figure 6: Software tools in CAS Mathematica.

In [31] the authors proposed the following new cumulative distribution function based on m existing ones:

$$G(t) = \frac{\sum_{k=1}^{m} F_k(t)}{m - 1 + \prod_{k=1}^{m} (F_k(t))^{\delta_k}}. \tag{13}$$

Based on the methodology proposed in the present note, the reader may formulate the corresponding modeling and approximation problems on his/her own.

Following the ideas given in [32] we propose the following new sigmoidal family with parameter λ:

$$\begin{aligned} G3(t) &= (1+\lambda)G(t) - \lambda G^2(t) \quad |\lambda| \in [0,1] \\ G(t) &= \tfrac{2F(t)}{1+F(t)} \\ F(t) &= \tfrac{1}{1+e^{-kt}} \\ G3(t_0^*) &= \tfrac{1}{2}; \quad t_0^* = \tfrac{1}{k} \ln \tfrac{1}{2(\lambda+\sqrt{1+\lambda^2})} \end{aligned} \tag{14}$$

In some cases the approximation of the shifted Heaviside function by $G3(t)$ is better in comparison to its approximation by $G(t)$

The generated sigmoidal function $G3(t)$ for $k = 20$ and $\lambda = 0.5$ is visualized on Fig. 7

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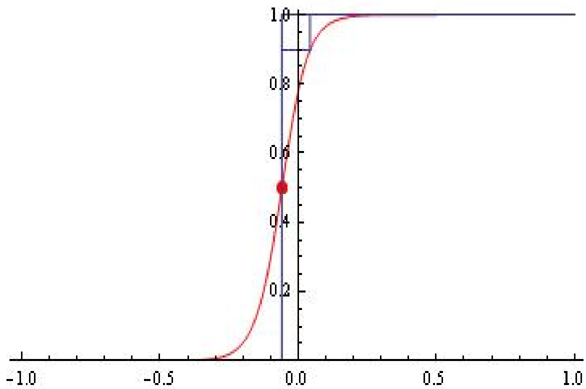

Figure 7: The H-distance between the functions $h_{t_0^*}$ and $G3(t)$ for $k = 20$ is $d = 0.102006$; $t_0^* = -0.058718$.

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