



A Mathematical Model of Solid Waste Accumulation and Treatment with a Varying Human Population Size

Isaac Rukundo, Pius Ariho

Department of Mathematics, Mbarara University of Science and Technology, P.O. Box 1410, Mbarara, Uganda

rukundo17@gmail.com  0000-0001-5359-2773

pariho@must.ac.ug  0000-0003-1861-4222

Abstract

Solid waste management has continued to be an increasing challenge worldwide and the situation has become worse in urban areas of developing countries. The rapid urban population growth, mainly due to high immigration and birth rates, has led to large amounts of solid waste, making it difficult for authorities to effectively manage the accumulated waste. Existing mathematical models of solid waste accumulation consider solid waste management by an external effort and do not address the contribution of the population in the management process. In this study, a mathematical model of solid waste accumulation is developed and analysed incorporating parameters for human immigration and solid waste recycling by particular population age groups. The solid waste is considered to be of two categories: biodegradable and non-biodegradable. Existence of equilibrium points is established and their stability analysed. Numerical simulations are done using MATLAB and Maple. Results show that solid waste increases with increasing human population and thus a solid waste free environment cannot be achieved. Sensitivity analysis suggests that improving the biodegradability of solid waste coupled with aiding solid waste decay and recycling reduces the final size of solid waste.

Keywords: Solid waste; Population growth; Recycling; Biodegradability

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1 Introduction

Solid waste is defined as any garbage or refuse and other discarded nonliquid and non-gaseous products from industrial, commercial, mining, agricultural, and other human activities [1, 6, 19]. Solid waste management is defined as an interrelated system of appropriate technologies and mechanisms involved in the generation, collection, storage, processing, transportation and disposal of solid waste at the lowest possible cost and risk to the health of the people and their environment [30]. It includes the whole process of generation, collection and treatment of solid waste. As noted by Henry *et al.* (2006) [14], Nemerow *et al.* (2009) [23] and Wilson *et al.* (2007) [37], solid waste management strategies are primarily meant to address health, environmental and economic concerns originating from inappropriate solid waste disposal [30]. These strategies may include converting solid waste into resources useful to households as well as business persons across the world [14, 32].

In most of the developing countries, solid waste management is mainly slowed down by limited resources, limited land for disposal purposes, opposition to proposed waste disposal sites and less participation of the population in solid waste eradication practices [10, 13]. The rapid population growth and rapid changes in lifestyle are responsible for the rapid increase in the quantity and changes in the composition of solid waste [27]. Spaargaren *et al.* (2006) [33], Oteng *et al.* (2013) [26] and Kinobe *et al.* (2015) [16] note that urban informal poor settlements are the most affected, with no or poor solid waste collection and management systems, where the uncollected waste is left lying on the ground for weeks. This deteriorates the environment and poses a public health risk. Thus, there exists a challenge to government authorities in ensuring adequate provision of waste management services to all residents [27] and this calls for the direct participation of the population in the process of solid waste minimization.

Globally, urban centers accumulate at least 1.3 billion tons of solid waste annually, and the amount is projected to rise to 2.2 billion tons by the year 2025 [24]. The high population growth rates together with the rural-urban migration that have escalated pose a big setback in the process of collecting and treating solid waste, especially in African urban areas [27]. In most of these urban areas, solid waste management is ultimately the responsibility of urban councils, while in most of the rural areas, the wastes are handled

at the household level [22]. According to the Uganda national population and housing census 2014-main report [34], the most commonly used method of solid waste disposal by households in Uganda is garden (44%) followed by burning (23%), local dump (11%), burying (8%), local urban supervised (7%), waste vendor (3%), and other methods (4%).

In Uganda, most of the towns such as Kampala, Arua and Mbarara are pressed with poor sanitation due to illegal disposal of faecal sludge in storm water drains, frequent sewer overflows and decreasing efficiency of existing wastewater treatment facilities [21]. In these towns, the main types of solid waste are domestic refuse, market refuse, commercial refuse, industrial refuse, abattoir wastes and hospital waste [21], and the solid waste collection and treatment is inadequate, illustrated by heaps of uncollected solid wastes along the road sides. In Kampala alone, about 28,000 tons of waste is collected and delivered to a landfill per month [17]. Kampala Capital City Authority (KCCA) is in charge of solid waste management in Kampala and records show that approximately 40% of the solid waste generated in the city is collected and delivered to a landfill [17]. The remaining uncollected waste is dumped indiscriminately along the streets, in river valleys and in drains, causing health and environmental challenges including flooding, breeding of insect vectors and housing rodent vectors [30, 35]. This results in the outbreak and spread of diseases such as Malaria, Cholera, Dysentery and Typhoid, among others [15, 16].

Solid waste can be categorised into biodegradable and non-biodegradable [25]. Biodegradable solid waste is decomposable and can be degraded by natural agents such as microbes and abiotic elements including temperature and ultraviolet light [8, 11]. Examples of biodegradable waste include human waste, green manure, paper waste, kitchen waste, slaughterhouse waste and biodegradable plastics [36]. Non-biodegradable waste refers to waste that cannot be decomposed or dissolved by natural agents [8, 11]. Most of the inorganic waste is non-biodegradable. It remains on earth for thousands of years without any degradation [8]. Examples include plastics, glass, batteries, cans, metals, and chemicals for agricultural and industrial purposes [29]. Processes such as oxidation and ozonation help to improve the biodegradability of some of the non-biodegradable waste materials. Thus pretreatment of such waste with oxidants helps in converting non-biodegradable waste to biodegradable waste [38].

Prediction of solid waste generation plays an important role in solid

waste management. Mathematical modelling provides an insight on the extent of the problem by providing data on the types of solid waste generated and their quantities. Developing a reliable model to predict the impact of population changes and solid waste recycling on solid waste accumulation would be a useful preliminary in the practice of solid waste management [12]. Existing mathematical models of solid waste accumulation consider an external effort in form of solid waste treatment with all solid waste in one class and this leaves unanswered decision questions of whether more effort should be put to biodegradable or non biodegradable waste and which effort is most suitable for a specific waste type.

Chen and Chang [9] developed a grey fuzzy dynamic model to predict solid waste accumulation in situations where there is limited data on solid waste generation and composition. The technique uses fuzzy goal regression as opposed to parameter identification in the grey model to ensure a better prediction accuracy. They provide a systematic estimation of accumulated waste in the absence of recycling with only incineration in place. Results indicate that solid waste accumulation would increase beyond the capacity of the incineration facility and therefore additional treatment and disposal approaches should be encouraged. However, it was found out that the dimension of the grey differential equation has a significant impact on the prediction accuracy, especially for models with higher dimension, in which case the accuracy would decrease if the dimension of the grey differential equations exceeds three.

Al-Khatib *et al.* [2] used system dynamics modeling to predict solid waste generation and associated disposal costs in developing areas, with Nablus city in Palestine as a case study. The study analyses the quantities and composition of generated waste by considering population and solid waste components such as metals and plastics together with the cost of recycling or disposing of the waste. Data analysis shows that solid waste accumulation increases with an increase in human population size. Comparative disposal cost analysis reveals that an increase in recycling would greatly reduce the solid waste disposal costs.

Senzige *et al.* [30] present a deterministic compartmental mathematical model to predict solid waste generation and treatment with population growth using ordinary differential equations. In this model, the solid waste accumulation Q has a threshold Q_m below which the solid waste amount is considered acceptable, and above which the external solid waste manage-

ment effort is triggered. The model assumes uniform waste generation rates and natural death rates for all population groups. Senzige and Makinde [31] present an improved version of [30] by considering different solid waste generation rates and natural death rates for each of the population groups. Both models [30] and [31] agree that solid waste increases with an increase in population. It was found out in [31] that as the solid waste accumulates, the applied effort increases leading to reductions in the solid waste accumulation, and in the long run, the accumulated waste balances with the treated waste.

All in all, existing models show that solid waste accumulation increases with population size and consider an external effort applied to reduce solid waste accumulation. However, the relationship between the waste reduction effort and the participating populations has not been given attention in the modelling process. This study presents a modified version of the Senzige and Makinde model [31] whereby biodegradable and non-biodegradable solid wastes are separated into two classes, and human effort in solid waste reduction is modelled as a linear function of the participating populations.

2 Model Formulation

The total human population size $N(t)$ is divided into three classes according to their age, that is, young (0-14 years), adults (15-64 years) and elderly (65+ years) denoted by $B_1(t)$, $B_2(t)$ and $B_3(t)$ respectively. Thus, the total population size at time t is given by $N(t) = B_1(t) + B_2(t) + B_3(t)$. Recruitment into the human population due to immigration occurs at a constant rate Λ , with τ_1 being the proportion of Λ that falls in age group B_1 , τ_2 the proportion of Λ that falls in age group B_2 and τ_3 the proportion of Λ that falls in age group B_3 . It is assumed that there is no emigration of the human population. The effective birth rate follows a logistic term $f(N) = r(1 - \frac{N}{K})$, where r is the human birth rate and K is the carrying capacity of the human population. It is assumed that only adults are capable of reproduction and that the effective birth rate depends on the ratio of the total population to carrying capacity. The young human population progress to the adult class at a rate p_1 , while the adults progress to the elderly class at a rate p_2 . The percapita natural mortality rates of the young, adults and elderly are μ_1 , μ_2 and μ_3 , respectively.

The human population at any time t accumulate solid waste of two

categories, that is, non-biodegradable solid waste and biodegradable solid waste [25] denoted by $Q_1(t)$ and $Q_2(t)$, respectively. Thus, the total solid waste accumulated at time t is $Q(t) = Q_1(t) + Q_2(t)$. It is assumed that solid waste is only accumulated through human activities. The rates at which the young, adults and elderly accumulate non-biodegradable solid waste are α_1 , α_2 and α_3 , respectively, while the rates at which the three groups accumulate biodegradable solid waste are γ_1 , γ_2 and γ_3 , respectively. Biodegradable solid waste decays naturally at a rate ε .

The non-biodegradable solid waste does not decay naturally in a short time as it takes thousands of years to decay. However, it is assumed that the non-biodegradable solid waste is exposed to natural oxidation which occurs at the rate η to improve its biodegradability, thus converting it into biodegradable waste. It is also assumed that solid waste recycling is done by only adults and the elderly at a uniform rate and that the activity needs more skills which the young may not have. Thus, biodegradable and non-biodegradable solid wastes are recycled at the rates ψ and δ , respectively. The model is illustrated by the compartmental diagram in Figure 1, with the parameters described in Table 1. From Figure 1, the model is defined by the following system of ordinary differential equations:

$$\begin{cases} \frac{dB_1}{dt} = \tau_1\Lambda + f(N)B_2 - p_1B_1 - \mu_1B_1; \\ \frac{dB_2}{dt} = \tau_2\Lambda + p_1B_1 - p_2B_2 - \mu_2B_2; \\ \frac{dB_3}{dt} = \tau_3\Lambda + p_2B_2 - \mu_3B_3; \\ \frac{dQ_1}{dt} = \alpha_1B_1 + (\alpha_2 - \delta)B_2 + (\alpha_3 - \delta)B_3 - \eta Q_1; \\ \frac{dQ_2}{dt} = \gamma_1B_1 + (\gamma_2 - \psi)B_2 + (\gamma_3 - \psi)B_3 + \eta Q_1 - \varepsilon Q_2, \end{cases} \quad (1)$$

where $f(N) = r(1 - \frac{N}{K})$, $\tau_1 + \tau_2 + \tau_3 = 1$, and all parameters are positive.

Theorem 2.1. *For nonnegative initial conditions $B_1(0) \geq 0$, $B_2(0) \geq 0$, $B_3(0) \geq 0$, $Q_1(0) \geq 0$ and $Q_2(0) \geq 0$, the solutions $B_1(t)$, $B_2(t)$, $B_3(t)$, $Q_1(t)$ and $Q_2(t)$ of System (1) are nonnegative for all $t > 0$.*

Proof. Solving the first three equations of System (1) leads to $B_1(t) \geq B_1(0)e^{-(p_1+\mu_1)t}$, $B_2(t) \geq B_2(0)e^{-(p_2+\mu_2)t}$ and $B_3(t) \geq B_3(0)e^{-\mu_3t}$. As $t \rightarrow \infty$, $B_1(t) \geq 0$, $B_2(t) \geq 0$ and $B_3(t) \geq 0$. For the last two equations of System (1), assuming that solid waste is accumulated at a rate greater or equal to the rate at which it is recycled, then $\max\{\alpha_1, \alpha_2, \alpha_3\} \geq \delta$ and $\max\{\gamma_1, \gamma_2, \gamma_3\} \geq \psi$, implying that $\frac{dQ_1}{dt} \geq -\eta Q_1$ and $\frac{dQ_2}{dt} \geq -\varepsilon Q_2$. Thus, $Q_1(t) \geq Q_1(0)e^{-\eta t} \geq 0$ and $Q_2(t) \geq Q_2(0)e^{-\varepsilon t} \geq 0$ for all time $t > 0$. \square

Table 1: Model parameters and their description

Parameter	Definition
Λ	Immigration rate into the human population
K	Carrying capacity of the human population
r	Human population birth rate
p_1	Progression rate of the young population to the adult class
p_2	Progression rate of the adult population to the elderly class
μ_1	Natural mortality rate of the young population
μ_2	Natural mortality rate of the adult population
μ_3	Natural mortality rate of the elderly population
α_1	Accumulation rate of non-biodegradable waste by the young
α_2	Accumulation rate of non-biodegradable waste by the adults
α_3	Accumulation rate of non-biodegradable waste by the elderly
γ_1	Accumulation rate of biodegradable waste by the young
γ_2	Accumulation rate of biodegradable waste by the adults
γ_3	Accumulation rate of biodegradable waste by the elderly
ε	Natural decay rate of the biodegradable waste
ψ	Rate at which biodegradable waste is recycled by humans
δ	Rate at which non-biodegradable waste is recycled by humans
η	Conversion rate of non-biodegradable waste to biodegradable waste

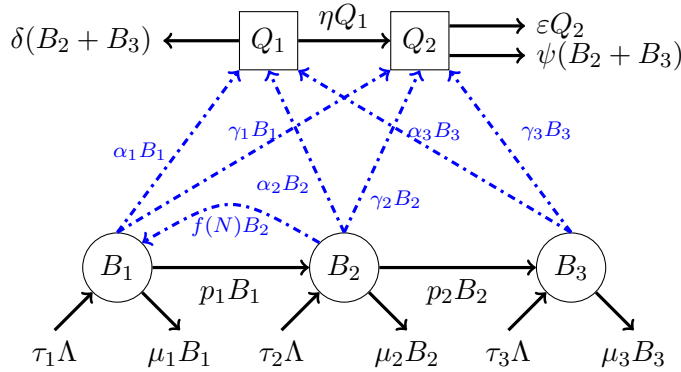


Figure 1: Compartmental diagram

Theorem 2.2. *The solutions $\{B_1(t), B_2(t), B_3(t), Q_1(t), Q_2(t)\} \in \mathbb{R}_+^5$ of System (1) are bounded.*

Proof. By definitions of N , Q_1 and Q_2 , and using equations of System(1), $\frac{dN}{dt} \leq \Lambda + r(1 - \frac{N}{K})B_2 - \mu N$ where $\mu = \min\{\mu_1, \mu_2, \mu_3\}$. For $N > K$, using the Comparison theorem, $\frac{dN}{dt} < \Lambda - \mu N$, with the solution $N(t) < \frac{\Lambda}{\mu} + (N_0 - \frac{\Lambda}{\mu})e^{-\mu t}$ where $N_0 = N(0)$. As $t \rightarrow \infty$, $N(t) < \frac{\Lambda}{\mu}$. For $N \leq K$, consider $\frac{dN}{dt}$ being negative, zero or positive with $N(0) = N_0 \leq K$. $\frac{dN}{dt} < 0$ implies that $N(t) < N_0 \leq K$ for all time $t > 0$. $\frac{dN}{dt} = 0$ implies that $N(t) = N_0 \leq K$ for all time $t > 0$. $\frac{dN}{dt} > 0$ implies that $N(t)$ increases and there exists a time $t_0 \geq 0$ at which $N(t) = K$ for the first time such that $\frac{dN}{dt}|_{t_0} = \Lambda - \mu K$. In this case, for all $t > t_0$, $N(t) < K$ if $\Lambda < \mu K$, $N(t) = K$ if $\Lambda = \mu K$, and $N(t) > K$ if $\Lambda > \mu K$ with the upper bound $\frac{\Lambda}{\mu}$ as in the previous argument. Thus, the solution set for the human population is contained in the region: $\Omega_N = \left\{ (B_1, B_2, B_3) \in \mathbb{R}_+^3 : 0 \leq N < \max\{K, \frac{\Lambda}{\mu}\} \right\}$. Similarly,

let $\alpha = \max\{\alpha_1, \alpha_2, \alpha_3\}$ and $N^* = \max\{K, \frac{\Lambda}{\mu}\}$. $\frac{dQ_1}{dt} < \alpha N^* - \eta Q_1$, with the solution $Q_1(t) < \frac{\alpha N^*}{\eta} + (Q_1(0) - \frac{\alpha N^*}{\eta})e^{-\eta t}$. As $t \rightarrow \infty$, $Q_1(t) < \frac{\alpha N^*}{\eta}$. Also, consider $\gamma = \max\{\gamma_1, \gamma_2, \gamma_3\}$. $\frac{dQ_2}{dt} < (\gamma + \alpha)N^* - \varepsilon Q_2$, implying that $Q_2(t) < \frac{(\alpha + \gamma)N^*}{\varepsilon} + (Q_2(0) - \frac{(\alpha + \gamma)N^*}{\varepsilon})e^{-\varepsilon t}$. As $t \rightarrow \infty$, $Q_2(t) < \frac{(\alpha + \gamma)N^*}{\varepsilon}$.

Thus, the solution set to System (1) is contained in the region

$$\Omega = \left\{ \left(\begin{array}{c} B_1 \\ B_2 \\ B_3 \\ Q_1 \\ Q_2 \end{array} \right) \in \mathbb{R}_+^5 \left| \begin{array}{l} 0 \leq N < N^* \\ 0 \leq Q_1 < \frac{\alpha N^*}{\eta} \\ 0 \leq Q_2 < \frac{(\alpha + \gamma)N^*}{\varepsilon} \end{array} \right. \right\},$$

in the nonnegative region of \mathbb{R}^5 . □

3 Model Analysis

3.1 Existence and stability of equilibria

Theorem 3.1. *Model System (1) has only one equilibrium and it is interior.*

Proof. System (1) is equated to zero and solved simultaneously to find the steady states as follows:

$$\begin{cases} \tau_1 \Lambda + r \left(1 - \frac{B_1^* + B_2^* + B_3^*}{K} \right) B_2^* - p_1 B_1^* - \mu_1 B_1^* = 0, \\ \tau_2 \Lambda + p_1 B_1^* - p_2 B_2^* - \mu_2 B_2^* = 0, \\ \tau_3 \Lambda + p_2 B_2^* - \mu_3 B_3^* = 0, \\ \alpha_1 B_1^* + (\alpha_2 - \delta) B_2^* + (\alpha_3 - \delta) B_3^* - \eta Q_1^* = 0, \\ \gamma_1 B_1^* + (\gamma_2 - \psi) B_2^* + (\gamma_3 - \psi) B_3^* + \eta Q_1^* - \varepsilon Q_2^* = 0. \end{cases} \quad (2)$$

Using the second and third equations of System (2),

$$B_1^* = \frac{(p_2 + \mu_2) B_2^* - \tau_2 \Lambda}{p_1} \quad \text{and} \quad B_3^* = \frac{\tau_3 \Lambda + p_2 B_2^*}{\mu_3}. \quad (3)$$

Substituting B_1^* and B_3^* in the first equation of System (2) leads to the quadratic equation:

$$a B_2^{*2} + b B_2^* + c = 0, \quad (4)$$

where

$$\begin{aligned} a &= (p_1 + p_2 + \mu_2) r \mu_3 + r p_1 p_2, \\ b &= (p_1 + \mu_1)(p_2 + \mu_2) \mu_3 K - r p_1 \mu_3 K - r \mu_3 \tau_2 \Lambda + r p_1 \tau_3 \Lambda, \quad \text{and} \\ c &= -(\tau_1 \Lambda p_1 + (p_1 + \mu_1) \tau_2 \Lambda) \mu_3 K. \end{aligned}$$

Notice that $a > 0$ and $c < 0$, and Equation (4) will always have two roots, one positive and one negative root, regardless of the sign of b . Since the negative root is biologically meaningless, we consider the positive root

$$B_2^* = \frac{-b + \sqrt{b^2 - 4ac}}{2a}. \quad (5)$$

Thus, from Equation (3), $B_1^* > 0$ whenever $(p_2 + \mu_2)B_2^* > \tau_2\Lambda$, and $B_3^* > 0$ always. Substituting B_1^* and B_3^* into the fourth and fifth equations of System (2) leads to

$$Q_1^* = \frac{(\alpha_3 - \delta)\tau_3\Lambda}{\mu_3\eta} - \frac{\alpha_1\tau_2\Lambda}{p_1\eta} + \left(\frac{\alpha_1(p_2 + \mu_2)}{p_1\eta} + \frac{\alpha_2}{\eta} + \frac{\alpha_3 p_2}{\mu_3\eta} - \frac{\delta(p_2 + \mu_3)}{\mu_3\eta} \right) B_2^* \quad (6)$$

and

$$Q_2^* = \frac{(\alpha_3 + \gamma_3 - \delta - \psi)\tau_3\Lambda}{\mu_3\varepsilon} - \frac{(\alpha_1 + \gamma_1)\tau_2\Lambda}{p_1\varepsilon} + \left(\frac{(\alpha_1 + \gamma_1)(p_2 + \mu_2)}{p_1\varepsilon} + \frac{\alpha_2 + \gamma_2}{\varepsilon} + \frac{(\alpha_3 + \gamma_3)p_2}{\mu_3\varepsilon} - \frac{(\delta + \psi)(p_2 + \mu_3)}{\mu_3\varepsilon} \right) B_2^*. \quad (7)$$

Thus, System (1) has only the interior equilibrium given by

$$E = (B_1^*, B_2^*, B_3^*, Q_1^*, Q_2^*). \quad \square$$

Theorem 3.2. *The interior equilibrium E is locally asymptotically stable.*

Proof. For the equilibrium E to be locally stable, all eigenvalues of the Jacobian matrix of System (1) evaluated at E must have strictly negative real parts (Theorem 3.1 of [20]). Evaluating the Jacobian at E gives:

$$J_E = \begin{bmatrix} C_1 & C_2 & -\frac{rB_2^*}{K} & 0 & 0 \\ p_1 & C_3 & 0 & 0 & 0 \\ 0 & p_2 & -\mu_3 & 0 & 0 \\ \alpha_1 & \alpha_2 - \delta & \alpha_3 - \delta & -\eta & 0 \\ \gamma_1 & \gamma_2 - \psi & \gamma_3 - \psi & \eta & -\varepsilon \end{bmatrix},$$

where

$$\begin{aligned} C_1 &= -\frac{rB_2^*}{K} - p_1 - \mu_1, \\ C_2 &= r - \frac{rB_1^*}{K} - \frac{2rB_2^*}{K} - \frac{rB_3^*}{K} \text{ and} \\ C_3 &= -p_2 - \mu_2. \end{aligned}$$

From $|J_E - \lambda I| = 0$, where I is a 5x5 identity matrix, we obtain the characteristic equation

$$(-\varepsilon - \lambda)(-\eta - \lambda) \begin{vmatrix} C_1 - \lambda & C_2 & -\frac{rB_2}{K} \\ p_1 & C_3 - \lambda & 0 \\ 0 & p_2 & -\mu_3 - \lambda \end{vmatrix} = 0, \quad (8)$$

which has two negative eigenvalues: $\lambda_1 = -\varepsilon$ and $\lambda_2 = -\eta$, and other three eigenvalues that satisfy the polynomial:

$$P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \quad (9)$$

with coefficients a_1 , a_2 and a_3 expressed in terms of a , b and c of Equation (4), where necessary, as follows:

$$\begin{aligned} a_1 &= \frac{r(-b + \sqrt{b^2 - 4ac})}{2aK} + p_1 + \mu_1 + p_2 + \mu_2 + \mu_3, \\ a_2 &= \frac{r(-b + \sqrt{b^2 - 4ac})}{2aK}(p_1 + p_2 + \mu_2 + \mu_3) \\ &\quad + (p_1 + \mu_1 + p_2 + \mu_2)\mu_3 + \frac{b + \sqrt{b^2 - 4ac}}{2\mu_3K}, \text{ and} \\ a_3 &= \frac{+\sqrt{b^2 - 4ac}}{K}. \end{aligned}$$

According to the Routh-Hurwitz criterion [3], the three eigenvalues of equation (9) have negative real parts if $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and $a_1a_2 - a_3 > 0$. Since $a > 0$ and $c < 0$, then a_1 , a_2 , a_3 and $a_1a_2 - a_3 > 0$ are always positive regardless of the sign of b . Thus, the interior equilibrium E is locally asymptotically stable. \square

Since E is the only equilibrium of System (1) and is locally asymptotically stable, we conjecture as follows.

Conjecture 3.3. *The interior equilibrium E is globally asymptotically stable.*

3.2 The special case where $\Lambda = 0$

In the absence of immigration, that is, $\Lambda = 0$, model (1) reduces to:

$$\begin{cases} \frac{dB_1}{dt} = f(N)B_2 - p_1B_1 - \mu_1B_1; \\ \frac{dB_2}{dt} = p_1B_1 - p_2B_2 - \mu_2B_2; \\ \frac{dB_3}{dt} = p_2B_2 - \mu_3B_3; \\ \frac{dQ_1}{dt} = \alpha_1B_1 + (\alpha_2 - \delta)B_2 + (\alpha_3 - \delta)B_3 - \eta Q_1; \\ \frac{dQ_2}{dt} = \gamma_1B_1 + (\gamma_2 - \psi)B_2 + (\gamma_3 - \psi)B_3 + \eta Q_2 - \varepsilon Q_2, \end{cases} \quad (10)$$

where $f(N) = r(1 - \frac{N}{K})$, $N(t) = B_1(t) + B_2(t) + B_3(t)$ and $Q(t) = Q_1(t) + Q_2(t)$.

The steady states of System (10) are obtained by solving Equation (4) with coefficients a , b and c evaluated at $\Lambda = 0$. The resulting equation $((p_1 + p_2 + \mu_2)r\mu_3 + rp_1p_2)B_2^{*2} + ((p_1 + \mu_1)(p_2 + \mu_2) - rp_1)\mu_3KB_2^* = 0$ has two solutions: $B_{2,1}^* = 0$ and $B_{2,2}^* = (rp_1 - (p_1 + \mu_1)(p_2 + \mu_2))\mu_3K / ((p_1 + p_2 + \mu_2)r\mu_3 + rp_1p_2)$. For $B_2^* = B_{2,1}^* = 0$, equations (3), (6) and (7) give $B_1^* = 0$, $B_3^* = 0$, $Q_1^* = 0$ and $Q_2^* = 0$ respectively. Thus the boundary equilibrium E_0 of system (10) is given by $E_0 = (0, 0, 0, 0, 0)$. $B_{2,2}^*$ can be re-written as

$$\frac{(p_1 + \mu_1)(p_2 + \mu_2)(\Gamma - 1)\mu_3K}{((p_1 + p_2 + \mu_2)\mu_3 + p_1p_2)r}$$

where

$$\Gamma = \frac{rp_1}{(p_1 + \mu_1)(p_2 + \mu_2)}. \quad (11)$$

Notice that $B_{2,2}^* > 0$ if and only if $\Gamma > 1$, and the interior equilibrium is $E_1 = (B_{1,2}^*, B_{2,2}^*, B_{3,2}^*, Q_{1,2}^*, Q_{2,2}^*)$ where:

$$\begin{aligned} B_{1,2}^* &= \frac{rp_1 - (p_1 + \mu_1)(p_2 + \mu_2)}{(p_1 + p_2 + \mu_2)\mu_3 + p_1p_2} \cdot \frac{(p_2 + \mu_2)\mu_3K}{rp_1}; \\ B_{2,2}^* &= \frac{rp_1 - (p_1 + \mu_1)(p_2 + \mu_2)}{(p_1 + p_2 + \mu_2)\mu_3 + p_1p_2} \cdot \frac{\mu_3K}{r}; \\ B_{3,2}^* &= \frac{rp_1 - (p_1 + \mu_1)(p_2 + \mu_2)}{(p_1 + p_2 + \mu_2)\mu_3 + p_1p_2} \cdot \frac{p_2K}{r}; \\ Q_{1,2}^* &= \left(\frac{\alpha_1(p_2 + \mu_2)}{p_1\eta} + \frac{\alpha_2}{\eta} + \frac{\alpha_3p_2}{\mu_3\eta} - \frac{\delta(p_2 + \mu_3)}{\mu_3\eta} \right) B_{2,2}^*; \end{aligned}$$

$$Q_{2,2}^* = \left(\frac{(\alpha_1 + \gamma_1)(p_2 + \mu_2)}{p_1 \varepsilon} + \frac{\alpha_2 + \gamma_2}{\varepsilon} + \frac{(\alpha_3 + \gamma_3)p_2}{\mu_3 \varepsilon} - \frac{(\delta + \psi)(p_2 + \mu_3)}{\mu_3 \varepsilon} \right) B_{2,2}^*.$$

If $\Gamma \leq 1$, $B_{2,2}^* \leq 0$ and E_0 exists as the only biologically meaningful equilibrium. In this case, the human population goes to extinction. Thus, system (10) has one boundary equilibrium and one interior equilibrium whenever $\Gamma > 1$, otherwise the system has only the boundary equilibrium.

Theorem 3.4. *The boundary equilibrium E_0 of System (10) is locally asymptotically stable whenever $\Gamma < 1$.*

Proof. The Jacobian matrix for System (10) at E_0 is given by

$$J_{E_0} = \begin{bmatrix} -p_1 - \mu_1 & r & 0 & 0 & 0 \\ p_1 & -p_2 - \mu_2 & 0 & 0 & 0 \\ 0 & p_2 & -\mu_3 & 0 & 0 \\ \alpha_1 & \alpha_2 - \delta & \alpha_3 - \delta & -\eta & 0 \\ \gamma_1 & \gamma_2 - \psi & \gamma_3 - \psi & \eta & -\varepsilon \end{bmatrix},$$

and its characteristic equation $|J_{E_0} - \lambda I| = 0$ gives three negative roots: $\lambda_1 = -\varepsilon$, $\lambda_2 = -\eta$, $\lambda_3 = -\mu_3$, and other two eigenvalues satisfy the equation $\lambda^2 + (p_1 + \mu_1 + p_2 + \mu_2)\lambda + (p_1 + \mu_1)(p_2 + \mu_2) - rp_1 = 0$ which can be re-written as $\lambda^2 + (p_1 + \mu_1 + p_2 + \mu_2)\lambda + (p_1 + \mu_1)(p_2 + \mu_2)(1 - \Gamma) = 0$. Using Routh-Hurwitz criterion [3], it can be observed that $(p_1 + \mu_1 + p_2 + \mu_2) > 0$, and $(p_1 + \mu_1)(p_2 + \mu_2)(1 - \Gamma) > 0$ whenever $\Gamma < 1$. Thus, E_0 is locally asymptotically stable whenever $\Gamma < 1$. \square

Theorem 3.5. *The interior equilibrium E_1 of System (10) is locally asymptotically stable whenever $\Gamma > 1$.*

Proof. Evaluating the Jacobian matrix of System (10) at

$$E_1 = (B_{1,2}^*, B_{2,2}^*, B_{3,2}^*, Q_{1,2}, Q_{2,2}^*)$$

yields

$$J_{E_1} = \begin{bmatrix} C_3 & C_4 & -\frac{rB_{2,2}^*}{K} & 0 & 0 \\ p_1 & -p_2 - \mu_2 & 0 & 0 & 0 \\ 0 & p_2 & -\mu_3 & 0 & 0 \\ \alpha_1 & \alpha_2 - \delta & \alpha_3 - \delta & -\eta & 0 \\ \gamma_1 & \gamma_2 - \psi & \gamma_3 - \psi & \eta & -\varepsilon \end{bmatrix}, \quad (12)$$

where $C_3 = -\frac{rB_{2,2}^*}{K} - p_1 - \mu_1$ and $C_4 = r - \frac{rB_{1,2}^*}{K} - \frac{2rB_{2,2}^*}{K} - \frac{rB_{3,2}^*}{K}$. It is clear from Matrix (12) that there are two strictly negative eigenvalues, that is, $-\varepsilon$ and $-\eta$. The remaining eigenvalues are investigated by considering the 3x3 submatrix

$$\begin{bmatrix} C_3 & C_4 & -\frac{rB_{2,2}^*}{K} \\ p_1 & -p_2 - \mu_2 & 0 \\ 0 & p_2 & -\mu_3 \end{bmatrix},$$

whose characteristic equation is of the form

$$A_1\lambda^2 + A_2\lambda + A_3 = 0 \tag{13}$$

where

$$\begin{aligned} A_1 &= \frac{(p_1 + \mu_1)(p_2 + \mu_2)(\Gamma - 1)}{(p_1 + p_2 + \mu_2)\mu_3 + p_1p_2} + p_1 + \mu_1 + p_2 + \mu_2 + \mu_3; \\ A_2 &= \frac{(p_1 + \mu_1)(p_2 + \mu_2)(\Gamma - 1)}{(p_1 + p_2 + \mu_2)\mu_3 + p_1p_2} \cdot (p_1 + p_2 + \mu_2 + \mu_3) \\ &\quad + (p_1 + \mu_1 + p_2 + \mu_2)\mu_3; \\ A_3 &= (p_1 + \mu_1)(p_2 + \mu_2)(\Gamma - 1)\mu_3. \end{aligned}$$

By Routh-Hurwitz criterion, the roots of Equation(13) have negative real parts if $A_1 > 0$, $A_2 > 0$, $A_3 > 0$ and $A_1A_2 - A_3 > 0$. It is clear that $A_1 > 0$, $A_2 > 0$, $A_3 > 0$ and $A_1A_2 - A_3 > 0$ whenever $\Gamma > 1$. Thus E_1 is locally asymptotically stable whenever $\Gamma > 1$. \square

Theorem 3.6. *The interior equilibrium E_1 of System (10) is globally asymptotically stable whenever $\Gamma > 1$.*

Proof. A Lyapunov function $L(B_1, B_2, B_3, Q_1, Q_2)$ is defined such that

$$\begin{aligned} L &= (B_1 - B_{1,2}^* + B_{1,2}^* \ln \frac{B_{1,2}^*}{B_1}) \\ &\quad + (B_2 - B_{2,2}^* + B_{2,2}^* \ln \frac{B_{2,2}^*}{B_2}) + (B_3 - B_{3,2}^* + B_{3,2}^* \ln \frac{B_{3,2}^*}{B_3}) \\ &\quad + (Q_1 - Q_{1,2}^* + Q_{1,2}^* \ln \frac{Q_{1,2}^*}{Q_1}) + (Q_2 - Q_{2,2}^* + Q_{2,2}^* \ln \frac{Q_{2,2}^*}{Q_2}). \end{aligned} \tag{14}$$

Upon differentiating Equation (14) with respect to t ,

$$\begin{aligned} \frac{dL}{dt} = & \left(\frac{B_1 - B_{1,2}^*}{B_1} \right) \frac{dB_1}{dt} + \left(\frac{B_2 - B_{2,2}^*}{B_2} \right) \frac{dB_2}{dt} + \left(\frac{B_3 - B_{3,2}^*}{B_3} \right) \frac{dB_3}{dt} \\ & + \left(\frac{Q_1 - Q_{1,2}^*}{Q_1} \right) \frac{dQ_1}{dt} + \left(\frac{Q_2 - Q_{2,2}^*}{Q_2} \right) \frac{dQ_2}{dt}. \end{aligned} \quad (15)$$

Substituting for the values of $\frac{dB_1}{dt}$, $\frac{dB_2}{dt}$, $\frac{dB_3}{dt}$, $\frac{dQ_1}{dt}$ and $\frac{dQ_2}{dt}$ in equation (15) yields

$$\frac{dL}{dt} = S - T,$$

where

$$\begin{aligned} S = & \frac{(B_1 + B_2 + B_3)rB_2}{K} \cdot \frac{B_{1,2}^*}{B_1} + (p_1 + \mu_1)B_{1,2}^* + p_1B_1 \\ & + (p_2 + \mu_2)B_{2,2}^* + (r + p_2)B_2 + \mu_3B_{3,2}^* \\ & + (\alpha_1B_1 + \alpha_2B_2 + \alpha_3B_3) + (\gamma_1B_1 + \gamma_2B_2 + \gamma_3B_3) \\ & + \left(\frac{Q_{1,2}^*}{Q_1} \delta + \frac{Q_{2,2}^*}{Q_2} \psi \right) (B_2 + B_3) + \eta(Q_{1,2}^* + Q_1) + \varepsilon Q_{2,2}^* \end{aligned}$$

and

$$\begin{aligned} T = & \frac{(B_1 + B_2 + B_3)rB_2}{K} + (p_1 + \mu_1)B_1 + p_1B_1 \cdot \frac{B_{2,2}^*}{B_2} \\ & + (p_2 + \mu_2)B_2 + \left(r \frac{B_{1,2}^*}{B_1} + p_2 \frac{B_{3,2}^*}{B_3} \right) B_2 + \mu_3B_3 \\ & + (\alpha_1B_1 + \alpha_2B_2 + \alpha_3B_3) \frac{Q_{1,2}^*}{Q_1} + (\gamma_1B_1 + \gamma_2B_2 + \gamma_3B_3) \frac{Q_{2,2}^*}{Q_2} \\ & + (\delta + \psi)(B_2 + B_3) + \left(1 + \frac{Q_{2,2}^*}{Q_2} \right) \eta Q_1 + \varepsilon Q_2. \end{aligned}$$

If $S \leq T$, then $\frac{dL}{dt} \leq 0$, noting that $\frac{dL}{dt} = 0$ if and only if $B_1 = B_{1,2}^*$, $B_2 = B_{2,2}^*$, $B_3 = B_{3,2}^*$, $Q_1 = Q_{1,2}^*$ and $Q_2 = Q_{2,2}^*$. Therefore, the largest compact invariant set in $\{(B_1, B_2, B_3, Q_1, Q_2) \in \Omega : \frac{dL}{dt} = 0\}$ is the singleton $\{E_1\}$, which is the interior equilibrium of System (10). Thus, by LaSalle's Invariance Principle [18], E_1 is globally asymptotically stable in Ω if $S < T$. \square

3.3 Effect of Λ , δ , η and ε on solid waste accumulation

From Equation (5), B_2^* can be rewritten as

$$B_2^* = \frac{-b(\Lambda) + \sqrt{b^2(\Lambda) - 4ac(\Lambda)}}{2a},$$

where $b(\Lambda)$, and $c(\Lambda)$ are linear functions of Λ . Thus Equations (6) and (7) give

$$Q_1^* = \frac{\xi_1(\delta)\Lambda}{\eta} + \frac{\xi_2(\delta)}{\eta} \left(\frac{-b(\Lambda) + \sqrt{b^2(\Lambda) - 4ac(\Lambda)}}{2a} \right)$$

and

$$Q_2^* = \frac{\xi_3(\delta, \psi)\Lambda}{\varepsilon} + \frac{\xi_4(\delta, \psi)}{\varepsilon} \left(\frac{-b(\Lambda) + \sqrt{b^2(\Lambda) - 4ac(\Lambda)}}{2a} \right),$$

where

$$\begin{aligned} \xi_1(\delta) &= \frac{(\alpha_3 - \delta)\tau_3}{\mu_3} - \frac{\alpha_1\tau_2}{p_1}, \\ \xi_2(\delta) &= \frac{\alpha_1(p_2 + \mu_2)}{p_1} + \alpha_2 + \frac{\alpha_3 p_2}{\mu_3} - \frac{\delta(p_2 + \mu_3)}{\mu_3}, \\ \xi_3(\delta, \psi) &= \frac{(\alpha_3 + \gamma_3 - \delta - \psi)\tau_3}{\mu_3} - \frac{(\alpha_1 + \gamma_1)\tau_2}{p_1} \text{ and} \\ \xi_4(\delta, \psi) &= \frac{(\alpha_1 + \gamma_1)(p_2 + \mu_2)}{p_1} + \alpha_2 + \gamma_2 \\ &\quad + \frac{(\alpha_3 + \gamma_3)p_2}{\mu_3} - \frac{(\delta + \psi)(p_2 + \mu_3)}{\mu_3}. \end{aligned}$$

Therefore,

$$\begin{aligned} Q^* &= Q_1^* + Q_2^* = \left(\frac{\xi_1(\delta)}{\eta} + \frac{\xi_3(\delta, \psi)}{\varepsilon} \right) \Lambda \\ &\quad + \left(\frac{\xi_2(\delta)}{\eta} + \frac{\xi_4(\delta, \psi)}{\varepsilon} \right) \left(\frac{-b(\Lambda) + \sqrt{b^2(\Lambda) - 4ac(\Lambda)}}{2a} \right). \end{aligned} \quad (16)$$

Since $\xi_1(\delta)$ and $\xi_2(\delta)$ decrease linearly with δ , and $\xi_3(\delta, \psi)$ and $\xi_4(\delta, \psi)$ decrease linearly with both δ and ψ , Equation (16) suggests that increasing δ and ψ reduces Q^* . It is clear that η , ε and Λ have a nonlinear relationship with Q^* .

4 Numerical Results

4.1 Estimation of parameter values

Based on the National Population and Housing Census of Uganda 2014 [34], the population was categorized in three age groups (0-14 years), (15-64 years) and (65+ years) corresponding to the age groups B_1 , B_2 and B_3 of the model. The total population of Uganda by mid 2019 was estimated to be 44.3 million people with the age groups (0-14), (15-64) and 65+ years taking 47%, 51% and 2% of the total population respectively [28], and 1.9 million people residing in Kampala city [4].

The birth rate of Uganda by mid 2019 was estimated at 0.037 births per person per year [28]. Considering a year of 365 days, this translates into a birth rate of $\frac{37}{365000}$ births per person per day. The progression rate of the population is calculated from the reciprocal of the class interval [31]. Thus considering population age classes of (0-14) years and (15-64) years, the young progress to the adult class at a rate of $\frac{1}{14}$ persons per year, which translates into $\frac{1}{5510}$ persons per day. In the same way, adults progress to the elderly class at a rate of $\frac{1}{49}$ persons per year, which translates into $\frac{1}{17885}$ persons per day. Uganda's life expectancy at birth is estimated at 63.5 years [34]. The natural mortality is calculated depending on the age group and life expectancy at birth [31]. Thus, the mortality rate is estimated at $\frac{1}{63.5}$ persons per year or $\frac{1}{23178}$ persons per day for the young population, and $\frac{1}{48.5}$ persons per year or $\frac{1}{17703}$ persons per day for the adult population. The mortality rate of the elderly is calculated based on the life expectancy at 60 years. Uganda's life expectancy at 60 years is 19.5 years [34]. Thus, the mortality rate of the elderly is estimated at $\frac{1}{14.5}$ persons per year, which translates to $\frac{1}{5293}$ persons per day.

Immigrants are estimated to constitute 0.1% of the total population [28] in a given year. Thus, Λ is estimated at 1900 individuals per year for Kampala city, translating to $\frac{1900}{365}$ persons per day. According to Senzige *et al.* (2014) [30], adults tend to migrate at a higher rate than the elderly. However, given the high birth rate of Uganda [34], and assuming that adults migrate with their children, the immigration rate of the young tends to be the highest. Thus τ_1 , τ_2 and τ_3 are approximated at 0.9, 0.07 and 0.03 respectively.

Aryampa *et al.* (2019) found out that the average waste generation rate in Kampala was 0.47 Kg/capita/day [4]. According to Banga (2013), 84% of

Table 2: Parameter estimates of the model

Parameter	Value	Source
K	10000	Assumed
Λ	1900/365	[28]
τ_1	0.9	Estimated §4.1
τ_2	0.07	Estimated §4.1
τ_3	0.03	Estimated §4.1
r	0.037/365	[28]
p_1	1/(14 x 365)	[31]
p_2	1/(49 x 365)	[31]
μ_1	1/(63.5 x 365)	[34]
μ_2	1/(48.5 x 365)	[34]
μ_3	1/(14.5 x 365)	[34]
α_1	0.30	[4]
α_2	0.28	[4]
α_3	0.26	[4]
γ_1	0.48	[4]
γ_2	0.45	[4]
γ_3	0.40	[4]
ε	0.072	Assumed
ψ	0.15	Assumed
δ	0.15	Assumed
η	0.0055	Assumed

the generated waste in Kampala is biodegradable [7]. Senzige and Makinde (2016) [31] argue that younger population groups generate waste at higher rates compared to the older population groups [31]. Thus, the solid waste generation rates are approximated at $\alpha_1 = 0.30$, $\alpha_2 = 0.28$, $\alpha_3 = 0.26$, $\gamma_1 = 0.48$, $\gamma_2 = 0.45$, $\gamma_3 = 0.4$, all measured in Kg per individual per day. The rest of the parameters ε , δ , η , ψ and K are assumed as seen in Table 2.

4.2 Simulations of the model

For simulation purposes, the initial conditions $B_1(0) = 10000$, $B_2(0) = 5000$, $B_3(0) = 1000$, $Q_1(0) = 100$ and $Q_2(0) = 100$ together with the pa-

parameters in Table 2 are used. With immigration, both young and adult populations increase rapidly and drop to a constant value as seen in Figure 2. The elderly population also rises steadily and approaches a fixed value. Figure 3 shows that both biodegradable and non-biodegradable solid waste quantities approach a constant nonboundary solution. Without immigration, all human populations go to extinction as illustrated in Figure 4, and the solid waste quantities Q_1 and Q_2 go to zero as shown in Figure 5. Notice that the parameter values in Table 2 give $\Gamma = 0.7389 < 1$ and thus Figures 4 and 5 illustrate the analytical results in which the boundary equilibrium E_0 is the only equilibrium of the model with $\Lambda = 0$ and $\Gamma < 1$. Since Γ increases with r , setting $r = \frac{0.098}{365}$ gives $\Gamma = 19.5719 > 1$, and in this case, both human population and solid waste accumulation never go to zero as illustrated in Figures 6 and 7. Thus, with $\Lambda = 0$ and $\Gamma > 1$, there is coexistence of human population and solid waste in agreement with the interior equilibrium E_1 .

Using Equation (16), a graph showing the equilibrium values of Q at different values of Λ and δ is obtained in Figure 8. It is observed that at very low levels of δ , Q^* increases with Λ . As δ gets larger, there exists an optimal value of Λ beyond which Q^* decreases. The effect of ψ on Q^* is similar to that of δ , except that the variations in Q^* resulting from changes in ψ are smaller compared to those resulting from similar changes in δ . Figure 9 shows that at very low nonzero values of ε , Q^* increases exponentially with Λ , reaches a peak value, and then decreases. η has a similar effect on Q^* as ε . For a given Λ , Q^* decreases exponentially with both η and ε , and their curves asymptotically approach values higher for ε than for η .

4.3 Sensitivity Analysis

The goal of sensitivity analysis is to decide qualitatively which parameters are most influential in the model output [5]. The sensitivity of the equilibrium value of the total accumulated solid waste Q^* (16) to the control parameters is determined using the normalized forward sensitivity index approach presented in [20] leading to the results in Table 3. Using the parameter values in Table 2, the resulting sensitivity indices in Table 3 show that δ , η , ψ , and ε have a reduction effect on Q^* . Since η and δ are the two most sensitive parameters, a slight increase in their values greatly reduces Q^* .

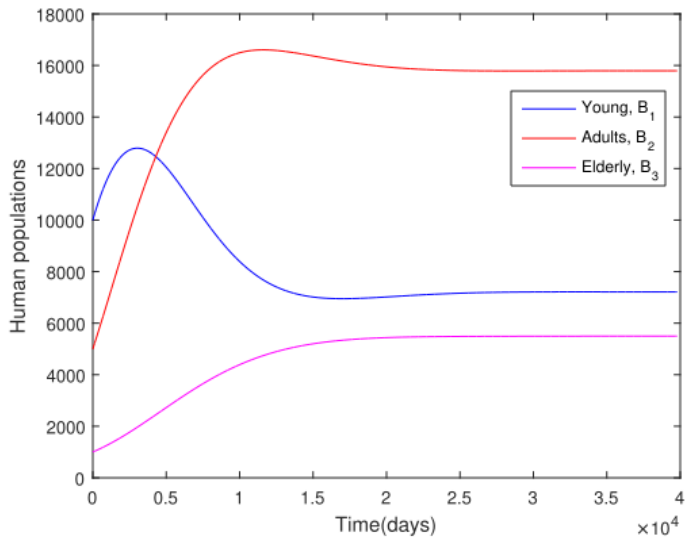


Figure 2: Population variations with time for $\Lambda \neq 0$

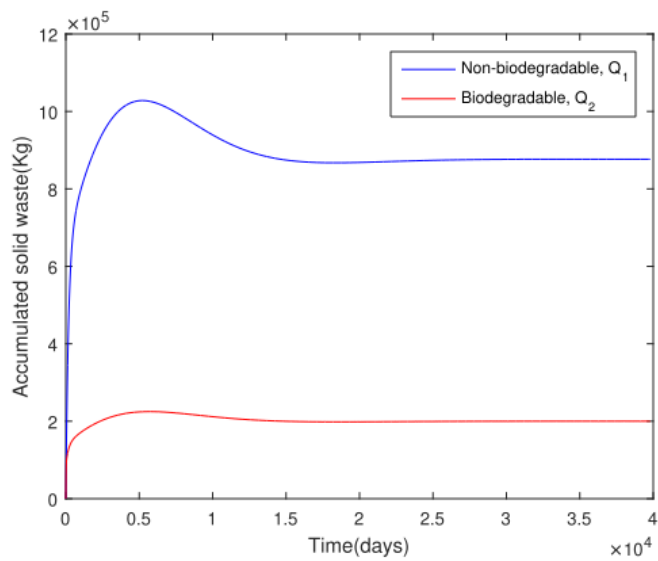


Figure 3: Solid waste variations with time for $\Lambda \neq 0$

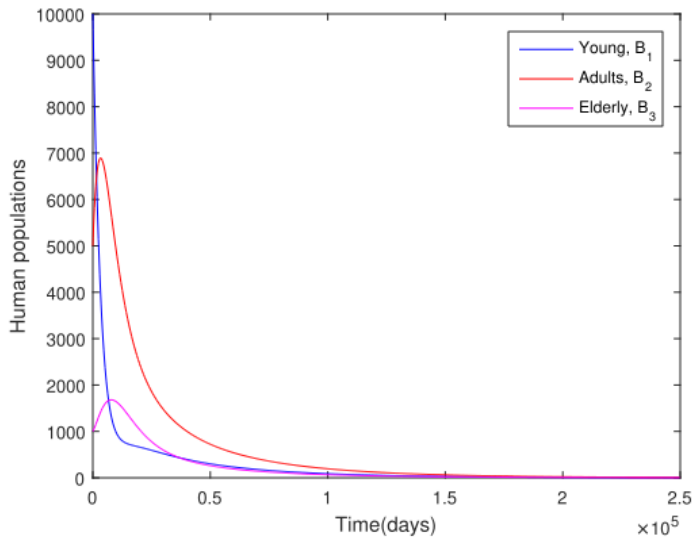


Figure 4: Population variations with time for $\Lambda = 0$ and $\Gamma < 1$

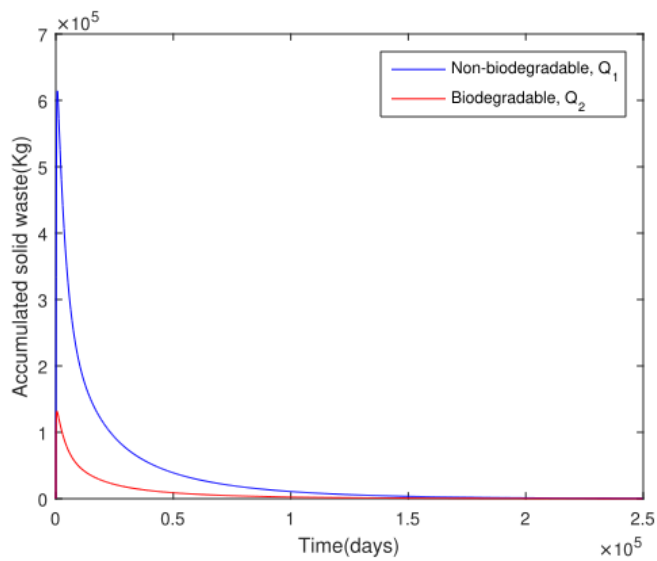


Figure 5: Solid waste variations with time for $\Lambda = 0$ and $\Gamma < 1$

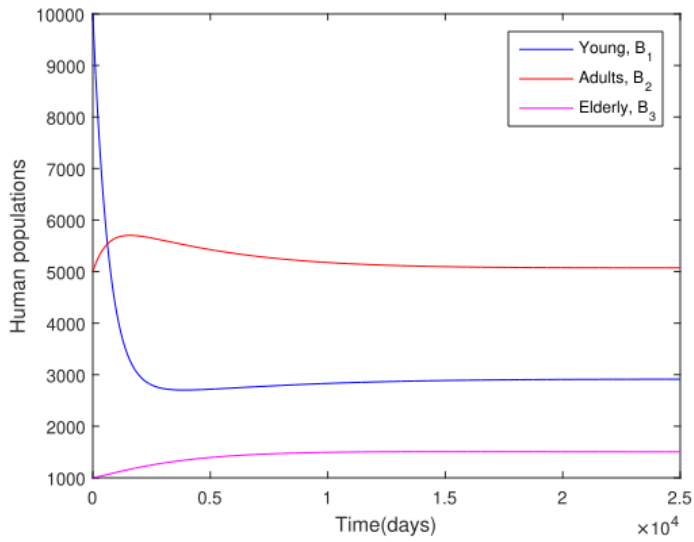


Figure 6: Population variations with time for $\Lambda = 0$ and $\Gamma > 1$

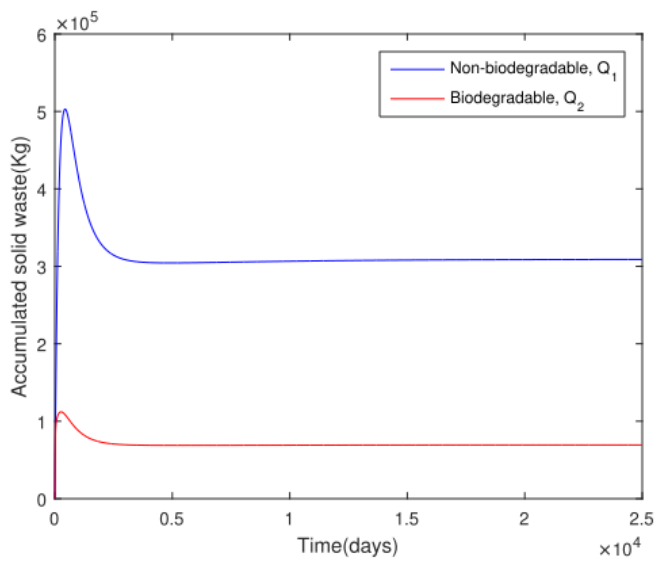


Figure 7: Solid waste variations with time for $\Lambda = 0$ and $\Gamma > 1$

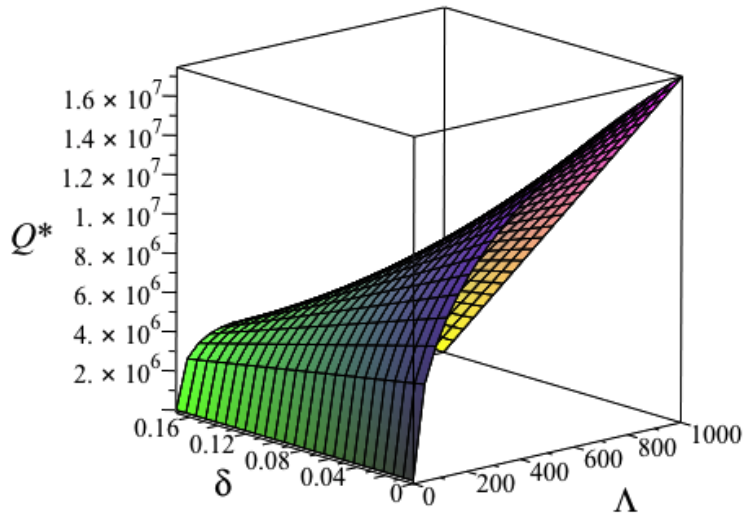


Figure 8: Q^* variations with δ and Λ

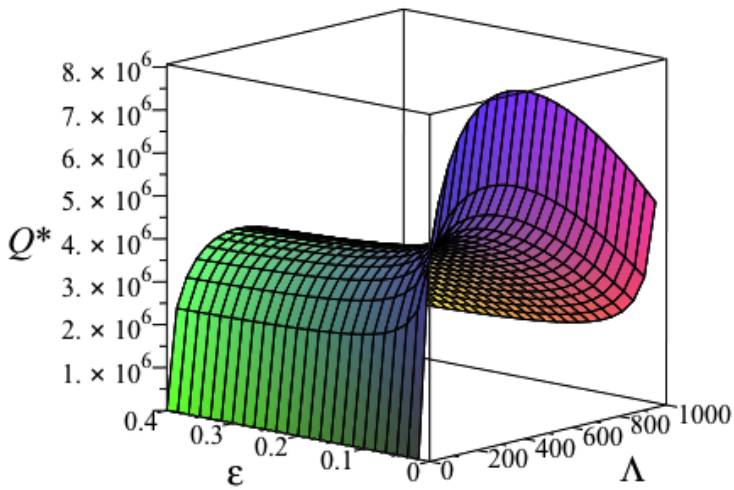


Figure 9: Q^* variations with ϵ and Λ

Table 3: Sensitivity index of Q^* to specific parameters

Parameter	Description	Index
δ	Recycling rate of Q_1	-0.58
η	Conversion rate of Q_1 to Q_2	-0.81
ψ	Recycling rate of Q_2	-0.04
ε	Natural decay rate of Q_2	-0.19

5 Discussion and Conclusion

In this study, a model of solid waste accumulation was developed and analysed. The model considered solid waste accumulation with varying human population size. Solid waste accumulation and recycling were modelled as linear functions of the contributing populations. Analysis of the model was done to assess the effect of immigration, birth rate and recycling on the final size of solid waste. It was discovered that with immigration, the model has only one interior equilibrium, and no other equilibria, implying that humans and solid waste coexist as shown in Figures 2 and 3. This contradicts the results from Senzige and Makinde [31] which suggest that in presence of constant recruitment in the human population, a waste free environment could still be achieved. The model has a special case for which there is no immigration, that is, $\Lambda = 0$, and the existence of equilibria depends on the value of a threshold parameter Γ . For $\Gamma \leq 1$, only the boundary equilibrium exists with both humans and solid waste going to zero as illustrated in Figures 4 and 5. For $\Gamma > 1$, there are two equilibria: one on the boundary and one in the interior of the feasible region as seen in Figures 6 and 7. Thus, the human population and solid waste coexist whenever $\Lambda > 0$ and when $\Gamma > 1$ for $\Lambda = 0$. Stability analysis was carried out, and it was found out that with immigration, the interior equilibrium is locally asymptotically stable. Numerical simulations suggest that the interior equilibrium is also globally stable. With no immigration, the boundary equilibrium is locally stable whenever $\Gamma < 1$. For $\Gamma > 1$, the boundary equilibrium is unstable, and the interior equilibrium is locally and globally asymptotically stable. Thus, the interior equilibrium is stable whenever it exists.

Numerical simulations in Figures 3, 5 and 7 show that the curve for

non-biodegradable waste rises above that of biodegradable waste. This is attributed to the faster decay rate of biodegradable waste compared to a slower process of converting non-biodegradable waste to biodegradable waste. Comparing Figures 3 and 7, the interior equilibrium values of biodegradable and non-biodegradable solid waste are relatively higher with immigration. The final size Equation (16) for solid waste suggests that for low recycling rates, increasing immigration increases the equilibrium value of solid waste. However, with a relatively large recycling rate, increasing immigration decreases the equilibrium value of solid waste as seen in Figure 8. Thus immigration should be coupled with skilling individuals with recycling knowledge. The threshold parameter Γ in Equation (11) is directly proportional to the birth rate r , and the special case with $\Lambda = 0$ undergoes a transcritical bifurcation when $r = (p_1 + \mu_1)(p_2 + \mu_2)/p_1$. Thus, for population control through birth rate, if there is no immigration, it is important to keep $r > (p_1 + \mu_1)(p_2 + \mu_2)/p_1$ to avoid extinction. From the sensitivity indices in Table 3, solid waste can also be checked by increasing the recycling rate, increasing the conversion rate of non-biodegradable waste Q_1 to biodegradable waste Q_2 and aiding the decay rate of biodegradable waste, in agreement with results from Al-Khatib *et al.* [2].

In conclusion, the dynamics of solid waste accumulation depend on immigration, birth rate of human population, solid waste recycling rate, the conversion rate of non-biodegradable waste to biodegradable state, and the solid waste decay rate. The effect of immigration on the equilibrium values depends on the recycling rate. Solid waste accumulation increases with birth rate and immigration, and decreases with recycling and decay. A waste-free environment is never achieved, and thus solid waste management strategies should target minimizing the final size of solid waste through recycling and improving the biodegradability or natural decay of solid waste.

6 Recommendations

The solid waste accumulation model suggests that it is not possible to attain a solid waste free environment. The model can be used as a policy tool in the prediction and management of solid waste. From the results of the model, it is recommended that:

- For populations with very low birth rates, immigration should be encouraged to avoid the possibility of extinction.

- For regions with no immigration, population growth largely depends on the birth rate, and this should be kept large enough to avoid extinction.
- Since a large population leads to large quantities of solid waste, community awareness should be carried out about appropriate solid waste disposal methods, including empowering individuals with the knowledge and skills of solid waste recycling.
- Given that biodegradable waste decays naturally, biological processes that accelerate the natural decay rate of organic matter should be encouraged.
- For non-biodegradable materials, policy makers should consider ensuring that the manufacturers set up recycling plants.

Data Availability

Parameter values used in this study were obtained from literature. Unavailable parameter values were estimated.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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