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A note on the piecewise smooth modified Schnute growth model. Applications

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Abstract

Following the ideas given in [13]–[15], in this article we study a hypothetical piecewise smooth modified Schnute growth function. Some numerical examples, using *CAS MATHEMATICA* are also given.

1 The Schnute's model. Introduction

Growth curves are found in a wide range of disciplines, such as biology, chemistry and medical science. Estimating the lag time in the growth process is a practically important problem [1], [2]. The lag time – t_{lag} is estimated by extending the tangent at inflection point to the initial baseline. The Schnute curve [3] is described by free parameters, each contributing to the characteristics of the curve: an initial lag or period of slow growth; a period of rapid exponential growth; a period of reduced growth rate.

The Schnute function finds applications in many scientific fields, including population dynamics, bacterial growth, population ecology, plant

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biology, chemistry and statistics. In his classical paper, Schnute [3] considered the model

$$L(t; l_1, l_2, a, b) = \left(l_1^b + \left(l_2^b - l_1^b \right) \frac{1 - e^{-a(t-t_1)}}{1 - e^{-a(t_2-t_1)}} \right)^{\frac{1}{b}}. \tag{1}$$

The values t_1 and t_2 are fixed and are normally taken to be the smallest and largest diameters in the data. $l_1 = L(t_1)$ and $l_2 = L(t_2)$ are the initial and final population densities, respectively (generally $l_2 > l_1$); $a \neq 0$ and $b \neq 0$ are rate parameters. For a visualization of this model at fixed values of the parameters: $a, b, t_1, t_2, l_1^b, l_2^b$, see Figs. 1–2.

For some details, see [4]–[10]. In his classical article, Schnute [3] considered the accelerated growth rate of species, and solved the model system:

$$\begin{aligned} \frac{dL}{dt} &= Lk, \\ \frac{dk}{dt} &= -k(a + bk), \end{aligned} \tag{2}$$

where parameters a and b are any constants.

Evidently

$$\begin{aligned} \frac{d^2L}{dt^2} &= \frac{dL}{dt}k + L\frac{dk}{dt} \\ &= \frac{dL}{dt}k - Lk(a + bk) \\ &= \frac{dL}{dt}k - \frac{dL}{dt}(a + bk) \\ &= \frac{dL}{dt}\left(-a + (1 - b)k\right). \end{aligned} \tag{3}$$

The basic form of this model is (1).

Example. We will observe the oil palm yield growth data.

The appropriate fitting of the experimental “oil palm yield data” [11], [12] by the Schnute growth function $L(t)$ with $a = 0.58$, $b = 0.015$, $t_1 = 5$, $t_2 = 17$, $l_1 = 18.43$, $l_2 = 38.45$ is visualized on Fig. 3.

The question of studying the degree of saturation of the classes of sigmoidal functions used in practice to the horizontal asymptote in the Hausdorff sense is extremely important. For example, the Hausdorff approximation [16] of the interval Heaviside step function $h(t)$ by the growth Schnute's model is discussed in [10].

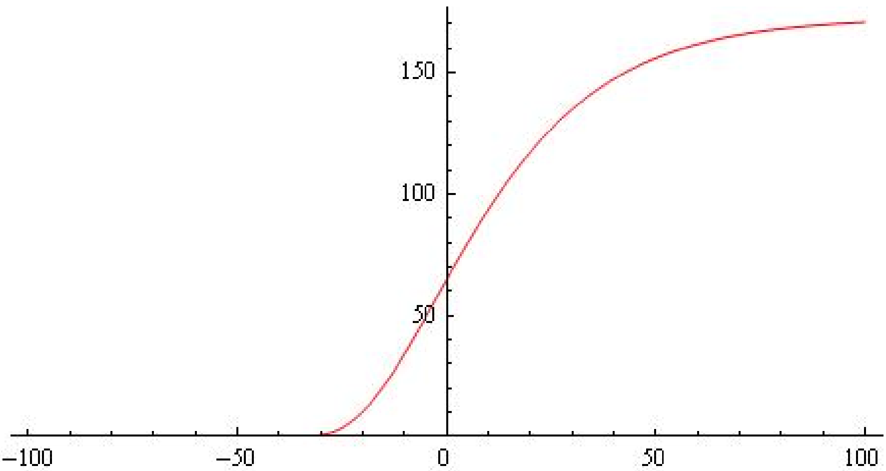

Figure 1: The function $L(t; l_1, l_2, a, b)$ for $a = 0.0435$, $b = 0.1146$, $t_1 = 0$, $t_2 = 129$, $l_1^b = 64.8$, $l_2^b = 172$.

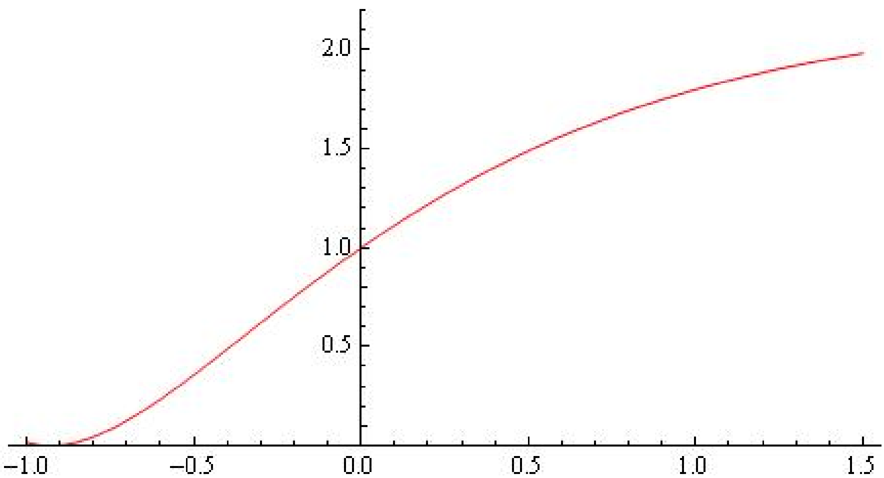

Figure 2: The function $L(t; l_1, l_2, a, b)$ for $a = 1.2$, $b = 0.5$, $t_1 = 0$, $t_2 = 1$, $l_1^b = 1$, $l_2^b = 1.8$.

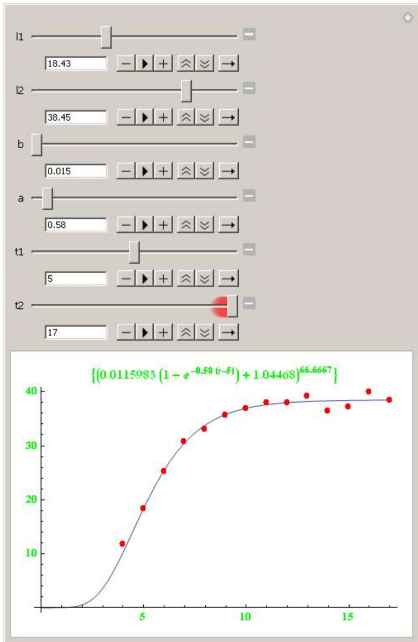

Figure 3: The appropriate fitting of experimental data by the Schnute growth function $L(t)$ with $a = 0.58$, $b = 0.015$, $t_1 = 5$, $t_2 = 17$, $l_1 = 18.43$, $l_2 = 38.45$ [10].

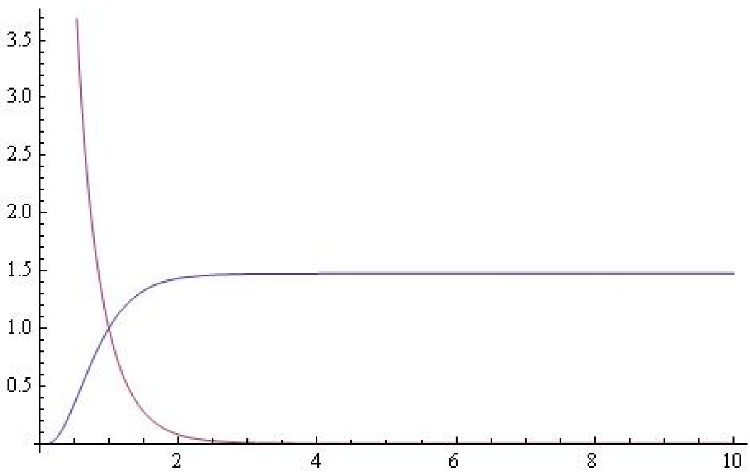

Figure 4: The solutions $L(t)$ and $k(t)$ for $a = 1.1$, $b = 0.5$ and $\kappa = 1$.

Remark. We will explicitly note that interesting modifications of the Schnute's method can be obtained. It suffices to consider the following hypothetical reaction network:

$$\begin{array}{lcl} L+K & \xrightarrow{\kappa} & L+L \\ K & \xrightarrow{a} & X \\ K+K & \xrightarrow{b} & Y \end{array} \tag{4}$$

wherein κ is the "rate constant". For example see Fig. 4 for solutions $L(t)$ and $k(t)$.

2 Another look at the modified Schnute's model

In this section is discussed an alternative investigation of an analogue of the Schnute's growth model.

Definition 1. The growth function $s_1(t)$ of the special Schnute model is defined for $t \geq 0$ and $A, B, k, r, b > 0$ by:

$$s_1(t) = B\left(\frac{1}{2} + A\left(1 - e^{-k(t-r)}\right)\right)^{\frac{1}{b}}, \tag{5}$$

for which $s_1(r) = B\left(\frac{1}{2}\right)^{\frac{1}{b}}$;

$$\lim_{t\to+\infty} s_1(t) = B\left(\frac{1}{2} + A\right)^{\frac{1}{b}} := B_1.$$

In the light of the discussions in this paper, the researcher can achieve saturation level

$$B_2 = \lim_{t\to+\infty} B\left(\frac{1}{2} + A\left(1 - e^{\frac{-k(t-r)}{1+k(t-r)}}\right)\right)^{\frac{1}{b}} = B\left(\frac{1}{2} + A(1 - e^{-1})\right)^{\frac{1}{b}}$$

if he uses, for example, the function

$$s_2(t) = B\left(\frac{1}{2} + A\left(1 - e^{\frac{-k(t-r)}{1+k(t-r)}}\right)\right)^{\frac{1}{b}}.$$

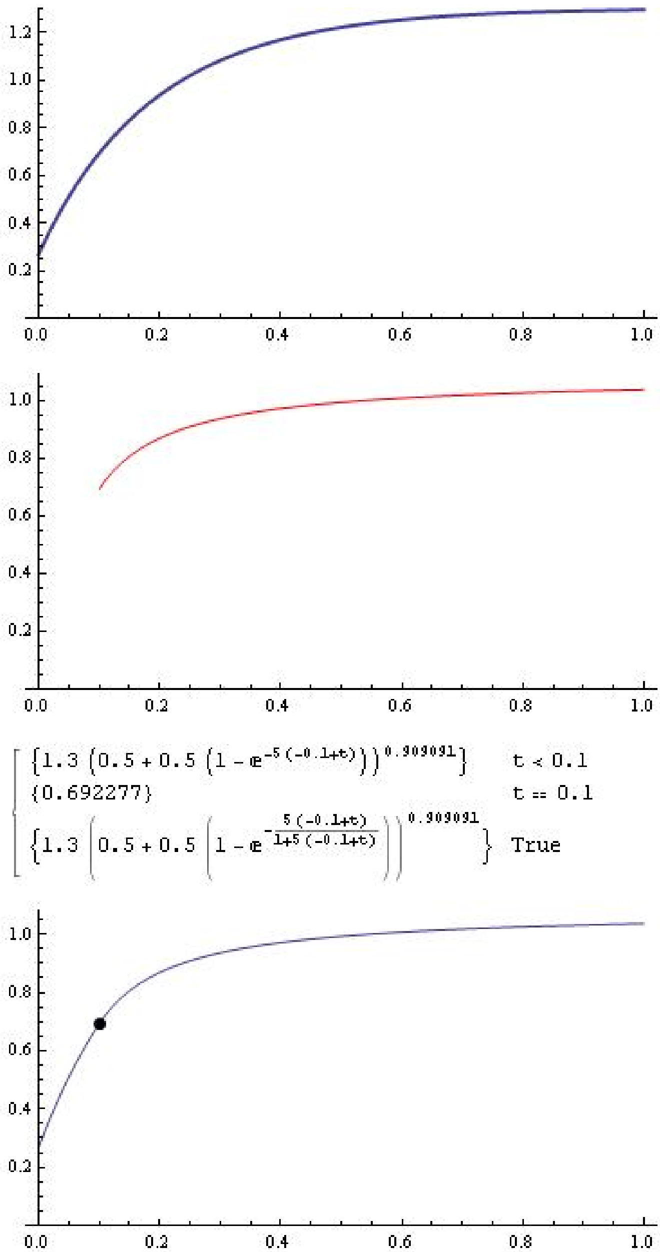

Figure 5: The functions $s_1(t)$, $s_2(t)$ and $S(s_1(t), s_2(t))$ for $B = 1.3$, $A = 0.5$, $r = 0.1$, $b = 1.1$, $k = 5$. Here $B_1 = 1.3$, $B_2 = 1.08066$.

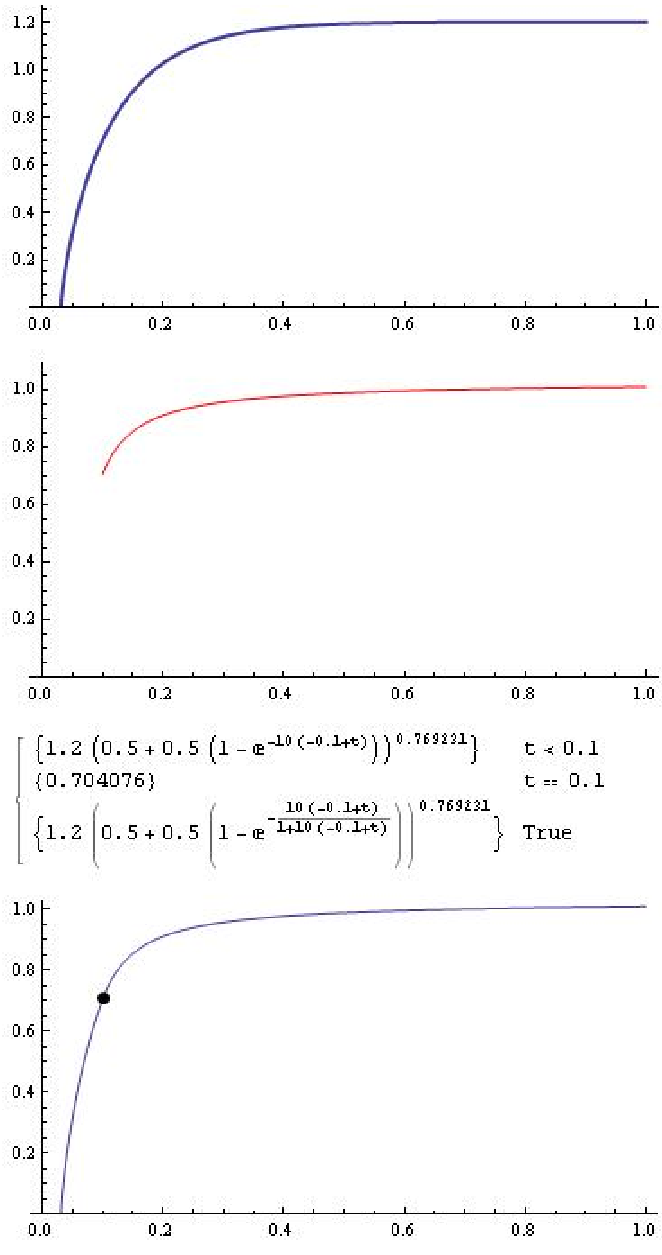

Figure 6: The functions $s_1(t)$, $s_2(t)$ and $S(s_1(t), s_2(t))$ for $B = 1.2,\ A = 0.5,\ r = 0.1,\ b = 1.3,\ k = 10$. Here $B_1 = 1.2,\ B_2 = 1.0263$.

It is easy to see that the hypothetical piecewise smooth Schnute growth model is of the form:

$$S(t) := \begin{cases} B\left(\frac{1}{2} + A\left(1 - e^{-k(t-r)}\right)\right)^{\frac{1}{b}} := s_1(t), \ \ t < r, \\ B\left(\frac{1}{2}\right)^{\frac{1}{b}}, \ \ t = r, \\ B\left(\frac{1}{2} + A\left(1 - e^{\frac{-k(t-r)}{1+k(t-r)}}\right)\right)^{\frac{1}{b}} := s_2(t), \ \ t > r. \end{cases} \tag{6}$$

Evidently,

$$s_1'(r) = s_2'(r).$$

The hypothetical piecewise smooth Schnute model $S(s_1(t), s_2(t))$ is depicted on Figs. 5–6. In addition, the reader can consider the interesting problem of approximating the Heaviside step function

$$h_r(t) = \begin{cases} 0, & \text{if} \quad t < r, \\ [0, B_2], & \text{if} \quad t = r, \\ B_2, & \text{if} \quad t > r, \end{cases}$$

with the new class of growth functions $S(s_1(t), s_2(t))$ with respect to the Hausdorff distance. In this regard, it is sufficient to use the methodology given in [17].

2.1 Application

For the normalized cumulative data [11]–[12]:

$$\begin{aligned} DataSchnute =& \{DataSchnute1 \cup DataSchnute2\} \\ DataSchnute1 :=& \{\{0.01, 0.05\}, \{0.02, 0.14\}, \{0.03, 0.17\}, \\ & \{0.05, 0.21\}, \{0.07, 0.3\}, \{0.1, 0.7\}\} \\ DataSchnute2 :=& \{\{0.1, 0.7\}, \{0.2, 0.89\}, \{0.8, 0.99\}, \\ & \{0.9, 0.995\}, \{1, 1.023\}\} \end{aligned}$$

we will use model $S(s_1, s_2)$:

$$s_1(t) = B\left(\frac{1}{2} + A\left(1 - e^{-k(t-r)}\right)\right)^{\frac{1}{b}} \quad 0 < t < r \approx 0.1$$

to approximate $DataSchnute1$ and

$$s_2(t) = B\left(\frac{1}{2} + A\left(1 - e^{\frac{-k(t-r)}{1+k(t-r)}}\right)\right)^{\frac{1}{b}}, \quad t > r$$

to approximate $DataSchnute2$.

For the actual data our model for

$$A = 0.5,\ k = 10,\ r \approx= 0.1,\ B = 1.2,\ b = 1.3$$

is depicted on Fig. 7.

The presented approach can be used successfully in the analysis of grouped data. For other research, see [18]–[24].

In conclusion, we will note that the modified Schnute model discussed in Section 2 and the disclosure of the intrinsic properties of this model, for example as a degree of saturation, may be useful in fitting a variety of data in the field of population dynamics.

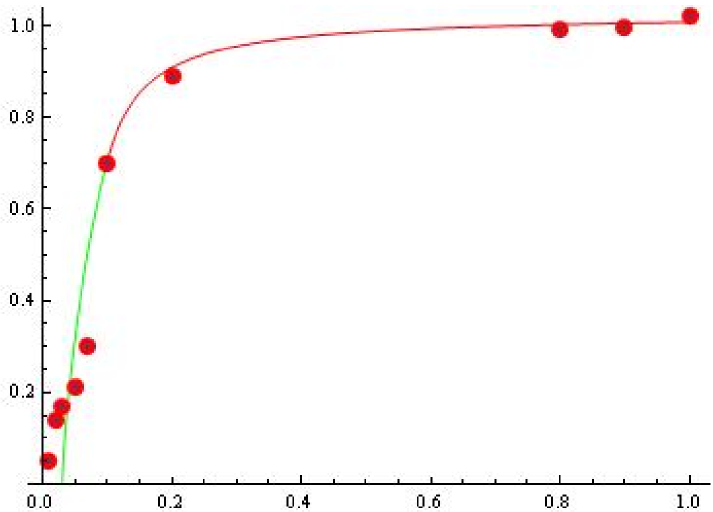

Figure 7: The functions $s_1(t)$–green and $s_2(t)$–red.

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